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THE RELATION OF THE EYES TO THE INTEGUMENTARY COLOR CHANGES IN THE CATFISH AMEIURUS

By G. H. Parker

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Communicated September 1, 1939

It is commonly stated in special papers and in monographs on chromatophoral activities that one eye is as efficient as two in the control of animal color changes. Such is the case, so far as I know, in the killifish, *Fundulus heteroclitus*. When one-eyed individuals of this species are closely compared with normal individuals, those with single eyes are found to change from pale to dark and the reverse in the same time and at the same rate as the normal fishes do. I have never been able to distinguish any significant color difference in two such lots of individuals. A remarkable example of the effect of a single eye on color changes was discovered by Pouchet (1877, 1880) in the trout which on the loss of one eye becomes dark contralaterally. This discovery was confirmed by von Frisch (1911) who attributed to abnormal conditions the inability of Steinach (1891) to obtain similar results.

The common fresh-water catfish, *Ameiurus nebulosus*, conforms in its one-eyed condition neither to the usual type of color response as exemplified in Fundulus nor to the exceptional one shown in the trout. In testing Ameiurus for chromatic reactions in relation to its eyes four classes of individuals were prepared: fishes from which both eyes had been removed, fishes from which the right eye had been excised, others from which the left eye had been taken and finally fishes in which both eyes were intact. Catfishes of all these four classes were subjected to darkness, to an illuminated black background and to an illuminated white one. The integumentary chromatophores in the catfish are all melanophores; these are either located in the epidermis and small in size (micromelanophores) or they rest on the underside of the derma and are relatively large (macromelanophores). In the color responses of this fish the conditions of both sets of melanophores

have been recorded. The approximate diameters of these color cells were measured under different conditions in thousandths of a millimeter and the cells chosen for these measurements were from the central part of the caudal fin at the region where the first branchings of the fin-rays occur.

Totally blind catfishes under all the environmental conditions noted maintain a coal-black tint indefinitely. The pigment in their melanophores is fully dispersed, giving these cells an apparent diameter of about 145 microns. The fine, thread-like branches of the micromelanophores in blinded Ameiurus are completely filled with pigment and the branches from one cell so combined with those from others as to form a finely meshed network. Each micromelanophore and its branches spread over a total area whose diameter is approximately 90 microns.

Normal fishes with both eyes functional and over an illuminated white background eventually took on a pale, translucent, greenish yellow tint. In such fishes the average diameter of the macromelanophores was found to be about 42 microns and that of the micromelanophores about 13 microns. Normal fishes over an illuminated black background became very dark though not coal-black. In such individuals the diameters of the macromelanophores averaged 136 microns and of the micromelanophores 82 microns. A well-developed micromelanophore network was present. The change from dark to pale in these fishes required about 32 hours and from pale to dark about 18 hours. One-eyed catfishes when first prepared turned very dark but never coal-black as did those fishes from which both eyes had been removed. The opposite sides of such fishes were of the same tint irrespective of which eye had been taken out. Measurements on catfishes which had lost the right eye yielded for the macromelanophores an average diameter of 98 microns and for the micromelanophores 31 microns, and for those that had lost the left eye 93 microns and 36 microns, respectively. When one-eyed pale catfishes were placed on an illuminated black background, they very slowly turned fully dark requiring some seven to eight days for the change instead of less than a day, the period taken by normal catfishes for such alterations in tint. When one-eyed dark catfishes were placed on an illuminated white background, they turned moderately pale in the course of a few days, but in my experience they never attained to full paleness and often returned to an intermediate tint or even to a fairly dark one. In such lengthy tests some of which were carried over several weeks, the fishes varied considerably being sometimes darker even up to full darkness and at other times paler but without reaching the full limit of this change. In a series of tests in which the optic nerve of one side of the head was severed instead of the removal of the eye of that side, the reactions of the catfishes thus unilaterally blinded were to all appearances the same as those of an individual blinded by enucleation. Thus there is no reason to suppose that the eye-ball of the

catfish gives out a hormone concerned in any direct way with the color changes of this fish.

It is evident from this account that a one-eyed catfish is neither like a one-eyed killifish which in its color changes follows exactly the normal two-eyed fish nor is it like the trout with its unilateral coloration. Ameiurus in this respect is unlike any other chromatic animal thus far described.

The work done on this fish in the last few years, part of which has already appeared (Parker, 1934; Abramowitz, 1936; Osborn, 1938; Wykes, 1938) and part of which is still in progress, indicates that there are two important receptors concerned with its color changes, the eyes and certain photoreceptors in its skin. It seems probable that through the action of light on the integumentary photoreceptors the pituitary complex is made to discharge intermedin whereby the fish is darkened. Through the eyes the environmental color conditions are met in that in black illuminated surroundings the dispersing melanophore nerve-fibres are excited thus supplementing the darkening effect of intermedin. White illuminated surroundings acting through the eyes excite the concentrating chromatophoral nerve-fibres and thus blanch the fish. There is also some evidence to show that the eye is concerned with the inhibition of the action of intermedin. The eye, however, seems not to have to do with the activation of the pituitary complex which appears to be excited only through the photoreceptors of the skin. From the standpoint of these results it is not surprising that eyeless catfishes are extremely dark irrespective of the white or black illuminated background on which they may rest, for with the loss of the eyes they appear to have lost the means of inhibiting pituitary activity and with the illumination of the skin this activity seems to be excited to the utmost.

In one-eyed catfishes, as already stated, the animals at first are dark and then after a day or more they become variable ranging in tint from considerable paleness to a strikingly dark tone. These diverse conditions are often exhibited more or less by the same individual fishes some of which keep their original dark tint while others take on a paler phase. It would appear as though the invariable darkness of the completely eyeless fish and the balanced color regulation of the normal individual gave way in the one-eyed forms to an imbalance of chromatic function characteristic of each specimen. In some the one eye appears to exert no check on the intermedin even in an illuminated white environment and the fish in consequence is very dark, but in others there is a vacillating and incomplete control of intermedin and other factors and the fish exhibits a state now dark, now pale, though never reaching complete paleness. It has been suggested that this fluctuating condition might be due to different degrees of irritation in the orbital wound, thus affecting possibly the cut end of the optic nerve. To test this I reopened and re-excited the orbital wounds in three darkish, one-eyed catfishes and in three pale ones. In none of these fishes, however,

did the general tint of their skins change through a test period of six hours. Hence I am not inclined to favor the view that variations in the state of the orbital wound are the occasion of the color differences in these one-eyed individuals. What causes such differences in tint I do not know.

Irrespective of the applicability of such general schemes to *Ameiurus nebulosus* as those already mentioned the fact remains that individuals of this species with one eye exhibit chromatic responses that conform neither to the general rule found for this condition in other animals nor to the peculiar type of response seen in the trout. Specimens of Ameiurus possessing only one eye appear to have a form of reaction peculiar to themselves, a form which may depend upon the large part played by intermedin in the chromatic changes of this fish as contrasted with the essential absence of intermedin from such animals as Fundulus.

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THE NATURE OF CHROMOSOME DIVISION AND THE DURATION OF THE NUCLEAR CYCLE

By A. Marshak*

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Communicated August 20, 1939

If each somatic chromosome in metaphase contains four chromonemata, the pairs of chromonemata disjoining in anaphase may be those which were originally separated at the onset of the preceding prophase[1,2] or in the second preceding prophase.[3,4] Thus if pairs of chromosome fragments in which each member of the pair contains only one chromonema are observed in the first anaphase following irradiation of the resting stage, they

may be of two types. On the first hypothesis they will arise from disjoining half-chromatids, and on the second from the half-chromatids going to the same pole. A preliminary study of such fragments offers evidence in support of the first hypothesis.

All the cells observed were from root-tips of six-day-old seedlings of *Vicia faba*, germinated on moist filter paper at 23°C. The methods used in treating and examining the material have previously been described.[5]

FIGURE 1

Right arrow points to a half-chromatid fragment which consists of a half turn of a single chromonema. The left arrow points to the chromosome or chromatid which lacks this segment. Magnification 2600 ×.

The fragments are occasionally filamentous, but are most frequently spherical and are never more than half the diameter of the anaphase chromosome or of the anaphase chromatid, using an alternative terminology.[7] Where the chromonemata of the normal chromosomes are well defined, these fragments never contain more than a portion of a single chromonema

(Fig. 1). Following Nebel,[6] they will be referred to hereinafter as minute fragments.

In table 1, column 6, the frequency of anaphase cells containing one or more minute fragments is given as per cent of the total anaphases observed after treatment with 122 roentgens. There is a very sharp maximun in frequency at about 18 hours after irradiation. This is to be contrasted with a maximum frequency for other types of abnormalities which come at about 3 hours after irradiation.[5] Furthermore, following a comparable dose of x-rays about 50 per cent of the anaphase cells show some type of chromosome abnormality at 18 hours after irradiation while only

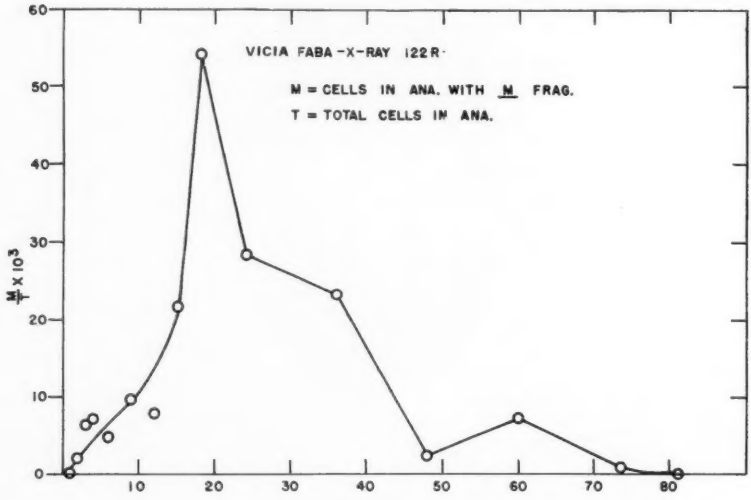


FIGURE 2

Minute fragments after treatment with 122 r of x-rays.

Ordinates—per cent cells with minute fragments × 10.
Abscissae—time in hours after irradiation.

5 per cent have minute fragments. When the dose is increased to 300 r the maximum frequency is about 13 per cent or a little more than twice that observed with 122 r.

With ordinary chromosome fragments 20 "n" units of neutrons give the same per cent abnormal anaphases as 130 r. Table 2 gives the frequency of minute fragments after treatment with 20 "n" of neutrons. There appears to be complete agreement not only in the time at which the maximum number of abnormalities is observed but also in the height of the peak as compared with the x-ray data (Figs. 2 and 3). However, in three other

experiments, using the same dose, the height of the peak was about 2 to 5 times as great (table 3). The significance of these fluctuations in the height of the peak is not understood. Since the temperature and the conditions during the neutron bombardment were kept constant, the fluctuations must be attributed to other factors. In all experiments the seeds were treated with calcium hypochlorite for half an hour, washed in distilled water and then soaked in distilled water for 24 hours, rinsed and transferred to filter paper in a moist chamber. In the earlier experiments the paper was soaked with distilled water, in the later ones with a 10 per cent dilution

TABLE 1

V. faba—m FRAGMENTS

SLIDE NO.	ROOTS	HOUR	TOTAL ANAPHASES	ANAPHASES WITH *M*	$M/T \times 10^2$
			122 r		
15/1	2	1	1095	0	0
2	2	2	975	2	0.205
3	3	3	909	6	0.66
78/1	4	4	282	2	0.71
2, 15/4	7	6	439	2	0.46
3, 15/5	7	9	824	8	0.97
81/1, 15/6	6	12	1832	14	0.77
81/2	2	15	462	10	2.16
81/3	2	18	648	35	5.41
4, 15/7	7	24	1487	42	2.82
5	2	36	258	6	2.32
6	4	48	806	2	0.23
7	2	60	414	3	0.73
15/8	4	74	1029	1	0.10
78/4	2	81	458	0	0
			300 r		
82/1	4	12	57	7	12.3
2	4	15	146	13	8.9
3	4	18	482	64	13.3
4	4	24	212	23	10.85
5	2	36	242	20	8.27

of Hoagland's solution. It seems possible that there may be some correlation between observed fluctuations and the differences in cultural conditions. An "n" unit of neutrons produces 6.6 times as many ordinary chromosome abnormalities as one roentgen of x-rays. Approximately the same ratio is observed for minute fragments when comparable culture conditions are maintained.

The series 106–111 were grown at the same time under the same conditions and seedlings selected at random were divided into two lots, one was given 20 "n," the other 50 "n." The maximum minute fragments observed with 20 "n" was 24.0 per cent, and with 50 "n" 58.9 per cent. The ratio

of the doses, 2.5, is about the same as the ratio of the effects, 2.45. This situation is similar to that observed with x-rays where the ratio of doses is 2.46 and of effects, 2.46. Although the data are scanty, they indicate that the per cent cells with minute fragments at maximum sensitivity varies linearly with dose, or that the log per cent cells without fragments is a linear function of the dose. If these fragments were produced by two independent ionizations, they would be expected to vary as the square of the dose. Since this does not appear to be the case, it follows that most of the fragments are produced by a single ion pair or cluster. In this connec-

TABLE 2

V. faba—*m* FRAGMENTS (INDIVIDUAL EXPERIMENTS)

SLIDE NO.	ROOTS	HOUR	TOTAL ANAPHASES	ANAPHASES WITH *M*	$M/T \times 10^2$
			20 "n"		
30/1	2	2	296	0	0
32/1	4	3	573	1	0.18
30/2	4	6	358	1	0.28
30/3	4	9	754	5	0.665
32/2	2	12	539	5	0.93
30/4	2	18	889	47	5.29
31/1	4	24	847	26	3.07
31/2	4	30	754	11	1.45
31/3	4	36	419	6	1.43
33/2	4	48	736	11	1.49
33/3	4	75	879	0	0
33/4	4	96	650	0	0
			50 "n"		
109/1	4	3	153	4	2.6
2	4	15	234	59	25.2
3	4	17	212	41	19.3
4	4	18	233	84	36.0
110/5	4	19	348	164	47.1
6	4	21	348	144	41.4
111/7	4	24	204	120	58.9
8	4	30	275	72	26.2
9	4	47	228	39	17.1

tion, it is significant that although a few of the fragments are non-terminal in origin, most of them seem to be terminal. We are left with the problem of why the fragments so frequently represent only short sections of the chromonema. Perhaps when long sections are involved, the half chromatids remain adherent and appear as single chromatid fragments.

Ordinary chromosome fragments form a small spherical body with apparently normal chromonemata when the rest of the nucleus goes into the resting stage. When the nucleus enters the prophase, these micronuclei become pycnotic and eventually disappear by anaphase. Some may per-

sist as spherical bodies which stain lightly and diffusely showing no chromonematic structure and are usually outside the spindle. The minute fragments must therefore have been initiated in the prophase or resting stage immediately preceding the anaphase in which they are observed, for otherwise they would have disintegrated by the second anaphase and would certainly not show any chromonematic structure. Furthermore if minute fragments arose from disintegrating micronuclei they should be seen as frequently as the large fragments at 3 hours after irradiation but are actually only one-tenth as frequent. Since the interval from prophase to anaphase takes only three hours, they must have been formed in the resting stage. After 48 hours subsequent to irradiation with light doses very few

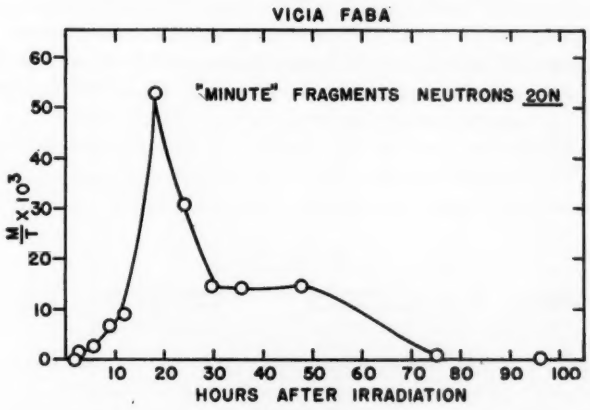


FIGURE 3

Minute fragments after treatment with 20 "n" of neutrons.

Ordinates—per cent cells with minute fragments × 10.
Abscissae—time in hours after irradiation.

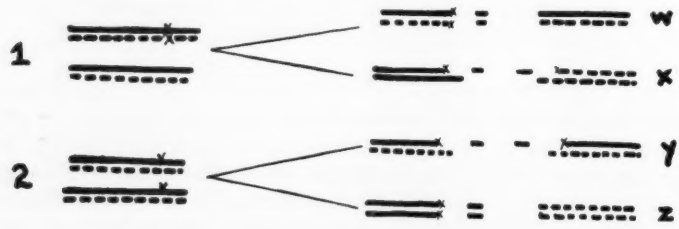
minute fragments or none at all are observed. Cells entering anaphase later than 48 hours after treatment must have been either in an early insensitive portion of the resting stage or in the preceding mitotic cycle at the time of irradiation. Since a considerable number of the cells in anaphase shortly after 48 hours contain disintegrating micronuclei the latter alternative seems the more likely. It follows from this that the maximum time from the beginning of the resting stage to anaphase after irradiation is approximately 48 hours. This includes the delay in nuclear development produced by the x-ray or neutron ionization. With heavier doses, the delay is greater as shown by the higher per cent minute fragments at 36–48 hours after treatment. By studying the variation in the time at which

the minute fragments disappear and the pycnotic micronuclei appear, it should be possible to determine the limiting value at a dose approaching zero, which would correspond to the normal duration of the unirradiated resting stage.

The sharp peak in the frequency of minute fragments at 18 hours after irradiation indicates that in the course of chromonematic development during the resting stage a condition is reached which is especially sensitive to x-ray and neutron ionization. Since the new chromonemata are not separated from the old till 15 hours later this sensitive condition probably corresponds to a critical stage in the synthesis of the new chromonemata.

Where the chromonemata were clearly defined, it was possible to classify the minute fragments. Of the unpaired fragments, 9 arose from the terminal portion of the half-chromatid, 2 were non-terminal in origin, and 6 were of indeterminable origin. Of the paired fragments 36 were derived from disjoining half-chromatids (*a*), 4 may have arisen from sister half-chromatids but overlapping made it impossible to eliminate the possibility that more than one chromatid was involved (*b*). The origin of 27 pairs could not be determined. In the last group 16 had the members of each pair completely free of each other and 11 were attached at one end. The conditions (*a*) and (*b*) are shown diagrammatically below.

If the old pair of chromonemata be represented by solid lines, the newly formed ones by dotted lines, and the alteration produced in the new or old chromonema by *x*, the methods by which types *a* and *b* might be produced are indicated in the following diagram.



Since the four chromonemata are visibly separated and approximately equidistant by metaphase, we may assume that the forces leading to the separation break the chromonemata in the regions previously altered by the ionization. The position of the fragments cannot then be used as an index of the behavior of chromonemata but the orientation of the deficient

strands from which they arose can. To obtain the condition indicated in *2*, two independent ionizations are necessary, since these strands are separated by distances of the order of one micron when they were exposed to the ionizing radiation. This is not in agreement with the observed linear relation of the frequency of the fragments to dose. In 1 the alteration involves a mother and daughter strand. At the time of exposure the two strands must have been close enough together to be altered by a single ion pair or cluster, a condition which is quite probable since there is no indication from microscopic examination or x-ray analysis that the strands

TABLE 3

V. faba—m FRAGMENTS (INDIVIDUAL EXPERIMENTS)
20 "n"

SLIDE NO.	ROOTS	HOUR	TOTAL ANAPHASES	ANAPHASES WITH *M*	$M/T \times 10^2$
107/2	4	15	496	71	14.3
3	4	17	520	80	15.4
5	4	19	388	79	24.0
6	4	21	465	79	17.0
7	4	24	363	24	6.6
8	4	30	664	27	4.1
106/9	5	48	678	16	2.4
97/1	2	3	251	9	3.58
2	2	17	251	33	13.15
3	2	18	371	39	10.5
4	2	19	350	38	10.85
5	2	21	240	16	6.66
22/1	2	1	786	0	...
2	2	2	349	0	...
3	4	3	523	0	...
4	4	6	312	0	...
23/1	4	9	626	8	1.28
2	4	12	490	10	2.04
3	4	205	95	10	11.58
24/1	2	26.2	298	16	5.70
2	2	30.	485	13	2.68
3	2	48.	268	7	2.61

separate till 15 hours later. If at anaphase a new and old strand go together to one pole, the condition $1w$ is expected. If the old pair is disjoined from the new pair, $1x$ is expected.

The fact that 36 were of type *a* and only 4 doubtful cases of type *b* offers conclusive evidence that the new pair of chromonemata which are synthesized in the resting stage are disjoined from that pair in the succeeding anaphase and not in the second succeeding anaphase. If the fragments of unknown origin and the four doubtful cases are considered as arising from sister chromonemata, there still is a somewhat greater number arising from

daughter strands. Since the latter are impossible on the second hypothesis, it seems necessary to rule that out in favor of the first.

I am deeply indebted to Dr. P. C. Aebersold for arranging and measuring the neutron exposures.

Summary.—Chromosome fragments containing a portion of only one chromonema are found after treatment of root tip of *Vicia faba* with neutrons or x-rays. There are very few during the first 12 hours and later than 48 hours after irradiation. At 18 hours a sharp maximum in frequency is observed which is attributed to a critical stage in the synthesis of new chromonemata in the "resting" nucleus. The maximum duration of the interval from the beginning of the resting stage to anaphase after treatment with 122 r or 20 "n" is probably about 48 hours. A method for determining the duration of the normal interval is indicated. Pairs of these fragments can be seen to be derived from disjoining half-chromatids, which is taken as evidence that the new pair of chromonemata formed in the resting stage are separated from the old pair in the succeeding anaphase.

* Fellow of the John Simon Guggenheim Memorial Foundation.

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A COMPARISON OF THE SENSITIVITY OF MITOTIC AND MEIOTIC CHROMOSOMES OF VICIA FABA AND ITS BEARING ON THEORIES OF CROSSING-OVER

BY A. MARSHAK*

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Communicated August 20, 1939

The sensitivity of the mitotic chromosomes in *Vicia faba* to x-rays has been determined.[1] It has also been shown that synthesis of the new chromonemata occurs in the nuclear resting stage.[2] Assuming that the sensitive chromonematic length remains the same, it is possible to predict the sensitivity of meiotic chromosomes in pachytene to be expected on the basis of different theories of chromosome pairing. The observed sensitivity gives good agreement with only one of these.

The per cent normal mitotic anaphases at the time when the maximum

number of abnormalities is observed (about three hours after irradiation) is an exponential function of the dose in roentgens of the form $Y = e^{-kx}$. For *Vicia faba* the slope of the curve is about 10.7×10^{-3}.[1] The chromosomes of the most sensitive stage of the first meiotic division (pachytene) of *Gasteria* also give an exponential curve and the cross-section of the x-ray-sensitive portion of the chromonema is of the same order of magnitude as that of *Pisum* and *Vicia*.[1] Direct comparisons, however, of mitotic and meiotic chromosomes have not been made with *Gasteria* since the seeds are not easily germinated. The sensitivity of meiotic chromosomes in pollen mother cells of *Vicia faba* was therefore determined.

The plants were oriented so that only the ends of the stalk bearing the flower buds were in the x-ray beam, and any large leaves overlying the buds

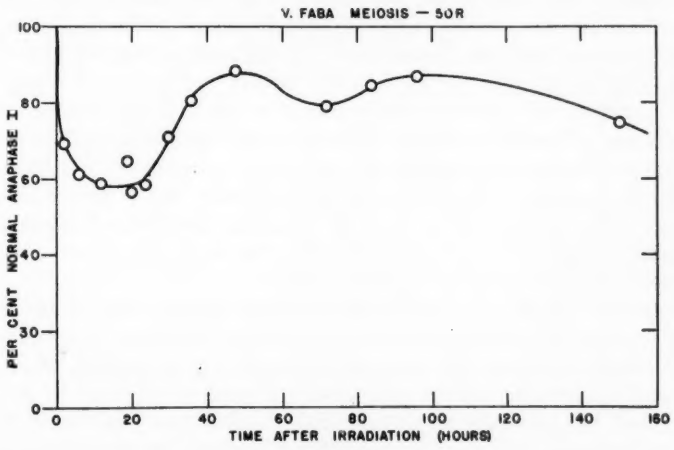


FIGURE 1

The per cent normal anaphase I as a function of time after treatment with 50 r.

were taped down out of the way. Scattering or absorption corrections were therefore negligible. The source of x-rays was a self-rectified, oil-cooled General Electric Company Maximar tube with a tungsten target, operated at 220 KVP and 15 ma.† The beam, filtered through 0.5 mm. Cu and 1 mm. Al, had a half value thickness of 1.25 mm. Cu. The material to be irradiated was 70 cm. from the target, a lead diaphragm with a 6×6 cm. aperture being placed at the end of the cone at 35 cm. All buds were fixed in a mixture of equal parts of glacial acetic acid and 95 per cent alcohol and were prepared for microscopic examination by smearing the buds in aceto-carmine.

Each anther has all of its pollen mother cells in approximately the same stage, but the anthers of any one bud are not all in the same stage of de-

velopment. It was therefore not possible to remove one anther from each bud and thus determine the distribution of stages irradiated as was done with *Gasteria*. However, it was found on examination of buds from a series of stalks that there was a fairly regular separation of about four buds between buds having one or more anthers in pachytene. For example, when the fourth or fifth bud of one inflorescence had pollen mother cells in pachytene, the first or second buds of the inflorescence immediately above it also had cells in pachytene. The distribution of first meiotic anaphases at twenty-four hours after irradiation parallels the distribution of pachytene stages a day earlier. It is therefore highly probable that these anaphase cells were in pachytene at the time of irradiation.

Plants were given 50 roentgens and examined at different intervals after irradiation. Only cells in the first meiotic anaphase were counted. They were classified into two categories, one in which the cells showed no chromosome abnormalities, the other containing cells with one or more chromosome attachments or fragments. The results are shown in figure 1. The maximum per cent abnormalities is observed at about 24 hours after irradiation. The most sensitive stage is thus the same in *Vicia* and *Gasteria*.[3] A high per cent of abnormalities is observed for a relatively long time, 18 hours, as compared with an interval of only about two or three hours in mitosis. The period from three to six days after irradiation has not been sufficiently explored to determine whether or not a second sensitive period is present as in *Gasteria*[3] or in *Tradescantia*.[4]

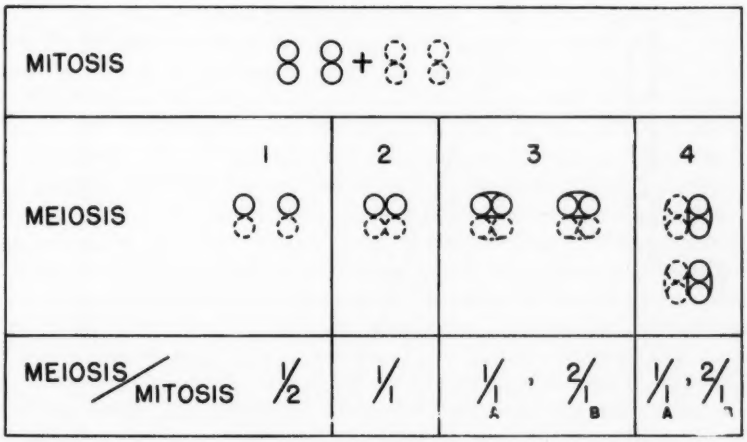
A series of plants were given different doses of x-rays and all the buds were fixed at 24 hours after irradiation. Counts of normal and abnormal first meiotic anaphases were made in the usual way. In figure 2 the logarithm of the per cent normal anaphases is plotted as a function of dose in roentgens. The per cent normal anaphases is clearly an exponential function of the dose of the same form as that observed with somatic chromosomes. The slope has the value 10.0×10^{-3} which may be considered the same within the limits of error of the experiment as the 10.7×10^{-3} obtained for the somatic chromosomes of *Vicia*.‡ Taking the density of air at 20°, 76 cm. Hg as 0.001205, the density of tissue as that of water at 20°C. (0.99823) and the charge on the electron as 4.77×10^{-10} e. s. u. the sensitive volume for the meiotic chromosomes will be

$$\frac{10 \times 10^{-3}}{828 \times 2.09 \times 10^{9}} = 5.78 \times 10^{-15} \text{ cm.}^3$$

For somatic chromosomes the sensitive volume is 6.18×10^{-15} cm.^{3}

The close agreement of the slopes of the sensitivity curves suggests that comparable conditions in relation to x-ray ionization exist in pachytene and the onset of the somatic prophase. In mitotic chromosomes the condition of maximum sensitivity may be attributed to the presence of two pairs

of chromonemata with the members of each pair within molecular distance of each other but with the pairs separated by much greater distances. Assuming that the x-ray-sensitive chromonematic length remains the same, the same sensitivity will be expected at pachytene if each bivalent at pachytene contains four chromonemata and these are equidistant and within molecular distance of each other.[2] If only homologous chromatids pair closely, the pairs remaining relatively far apart, only half the somatic sensitivity is to be expected.[1] However, if close association occurs between pairs and if each chromatid is twice as broad or contains two chromonemata the same sensitivity as in mitosis is to be expected.[3,4] These conditions are shown diagrammatically below.



In *a* diagonal fusions between homologous half-chromatids are not allowed but are allowed in *b*. Pachytene separation is in the direction of the top and bottom of the page.

In condition *4*, if abnormalities can be formed by attachments diagonally between homologous half-chromatids the sensitivity will be twice that of somatic chromosomes. If structural conditions prevent diagonal adhesions, the mitotic sensitivity is expected.

From the x-ray data alone, one may choose between the four situations only on the basis of agreement of meiotic and mitotic sensitivities, which is obtained with conditions *2*, *3a* and *4a*. In the absence of further x-ray evidence, selection or elimination of either of these possibilities must be made on the basis of other existing cytological and genetical data. If the separation between strands seen at pachytene in some species (e.g., *Zea mays*) is between pairs of synapsed chromatids, condition *4* is a probable one; but if the separation is between pairs of sister strands, i.e., separation

along the plane of synapsis, *4* becomes improbable. The behavior of the knobbed chromosomes of *Zea* indicates that the latter method of separation occurs.[5]

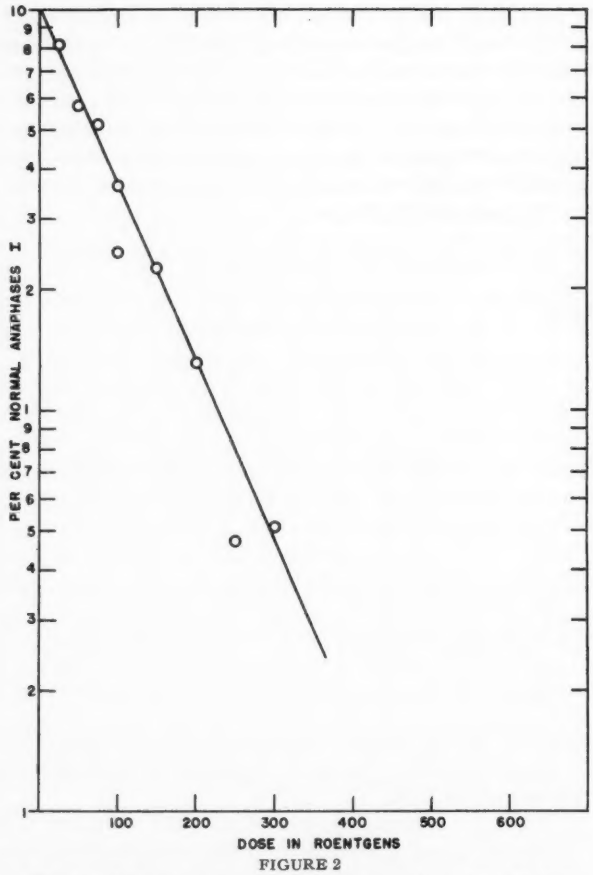


FIGURE 2

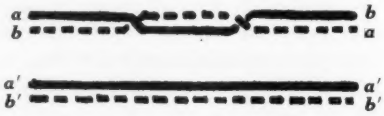
The logarithm of the per cent normal anaphase I at 24 hours after treatment with different doses of x-rays.

The observation of a high frequency of abnormalities over a period of about 18 hours suggests that the condition of high sensitivity exists for a period of that duration or that the time taken to reach first anaphase after synapsis may vary by as much as 18 hours. The latter alternative seems rather unlikely from the following consideration. Not only in different species of plants but in insects as well,[6,7] where it is difficult to imagine conditions external to the chromosomes as being the same, the time at which

the maximum abnormalities are observed is about 24 hours after irradiation. If under such different external conditions, the time from pachytene to first anaphase is approximately the same, it appears unlikely that relatively slight variations in the external environment of the same species will produce the large time variations needed in this case. The problem needs further investigation before a definite conclusion is reached, but for the time being it does not seem unjustified to accept the second alternative. High sensitivity must then be maintained throughout the entire period from synaptene to the end of pachytene. If the pairs of homologous chromatids become visibly separated in pachytene as in *Zea*, schemes *1* and *4* would have to be eliminated. In scheme *2* it is difficult to see why sister strand crossing-over should not occur and genetic data make this highly improbable. Only *3* is left as an adequate condition. If in *3* the chromatids had already divided, crossing-over between half-chromatids could occur. Discarding this as an impossible condition, each chromatid must be single and yet have a sensitive volume twice as great as that of the somatic chromatid. This condition would obtain if each chromatid had already synthesized a new strand which had not separated from the mother strand, a condition which has been shown to exist in the mitotic resting nucleus.[8] Recapitulating, the sequence of events is as follows: There are two chromonemata in each chromosome of the somatic resting nucleus. Each of these pairs of chromonemata synthesizes a new pair which in mitosis becomes separated from the mother pair at the end of the resting stage. The resting stage preceding meiosis is not terminated in this manner; instead the new and old strands remain together and pair with the chromatids of their homologues. In pachytene the pairs of homologous chromatids separate and eventually, sometime after the end of pachytene, the half-chromatids separate. Darlington's theory of crossing-over requires that division of the synapsed homologues precede crossing-over.[9,10] If that theory be applied to the present system, crossing-over would take place between half-chromatids. The evidence is therefore against Darlington's theory.

On the basis of scheme *2a* some of the known facts concerning crossing-over could be accounted for. Crossing-over would take place during synapsis and early pachytene. Sister strand crossovers would not occur since there is no molecular approximation of these strands. There would also be no half-chromatid crossovers. There would be a positive correlation between the frequency of crossovers and chiasmata since chiasmata would be formed during the pachytene separation at each point of crossing-over, e.g.:

Interference in any one pair of chromatids may occur if twisting of the homologous chromatids is prerequisite to crossing-over:



This would not account for interference in the other pair of chromatids, but if alternate pairing takes place, e.g., *a* pairs with *b* at one level, then with *b′* at the next, the interference is effectually transferred to the other pair.

Since the synapsed strands come to within molecular distances of each other, crossing-over would likewise have to be put on a molecular basis, and a truly adequate theory of crossing-over should define the molecular forces involved. We must therefore consider that the mechanism of crossing-over still remains unexplained and may have to remain so until the molecular structure is understood. One simple consequence of the postulated condition at synapsis may prove useful. This is that synapsis must take place between other faces of the sensitive portion of the chromonemata than the ones along which the new strands are synthesized.

Summary.—The maximum number of chromosome abnormalities is observed at about 24 hours after irradiation with x-rays. The most sensitive stage in the meiotic cycle is pachytene. The sensitivity curve for meiotic chromosomes has almost exactly the same slope as that for somatic chromosomes. The relation of these observations to theories of crossing-over is discussed.

* Fellow of the John Simon Guggenheim Memorial Foundation.

† The laboratory is indebted to the General Electric Company for the use of this x-ray equipment.

‡ Sax considers that in *Tradescantia* meiotic chromosomes are more sensitive to x-rays than mitotic chromosomes, but he gives no quantitative comparisons.[4]

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[3] Marshak, A., *Jour. Gen. Physiol.*, **19**, 179–198 (1935).
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[9] Darlington, C. D., *Jour. Genet.*, **31**, 185–212 (1935).
[10] Darlington, C. D., *Recent Advances in Cytology*, p. 549, P. Blakiston's Sons and Co., Philadelphia (1937).

THE MAGNETIC PROPERTIES OF THE COMPOUNDS ETHYLISOCYANIDE-FERROHEMOGLOBIN AND IMIDAZOLE-FERRIHEMOGLOBIN

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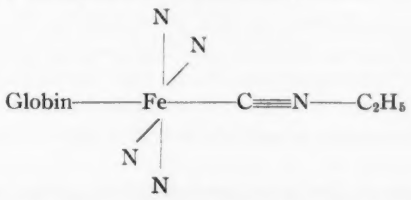
In the course of the series of magnetic investigations of hemoglobin being carried out in these Laboratories we have studied two new compounds, ethylisocyanide-ferrohemoglobin and imidazole-ferrihemoglobin, obtaining the results reported below.

Ethylisocyanide-ferrohemoglobin.—The compounds oxyhemoglobin, carbonmonoxyhemoglobin,[1] nitric oxide hemoglobin[2] and ferrohemoglobin-cyanide ion[3] have been shown to be diamagnetic, from which it is inferred that the iron atoms of the heme groups form octahedral covalent bonds in these molecules. Another compound of ferrohemoglobin, that with methyl isocyanide, has been reported by Warburg, Negelein and Christian,[4] and it would be of interest to determine from its magnetic properties whether or not the bonds formed by the iron atoms in this molecule also are of the octahedral covalent type. It is probable that similar compounds are formed with all of the lighter alkyl isocyanides, and we have found it convenient to study the reaction of ferrohemoglobin with ethyl isocyanide rather than that with the methyl derivative.

Ethyl isocyanide combines easily with ferrohemoglobin to give a compound with an absorption spectrum consisting of two narrow bands, at about 5540 and 5250 A.; this spectrum resembles closely those of the other compounds of ferrohemoglobin. Ethylisocyanide-ferrohemoglobin is completely diamagnetic, and accordingly contains no electrons with unpaired spins. The substance is shown by the magnetic titration described in the experimental part to have the ratio isocyanide to heme iron equal to unity. It is probable that the bond to the isocyanide molecule connects the iron and carbon atoms, corresponding to the structure



This structure is analogous to those proposed for the other ferrohemoglobin compounds.

Imidazole-ferrihemoglobin.—The interaction of imidazole and hemoglobin is of interest because of the likelihood that the bond between heme iron and globin involves the imidazole group of the side-chain of a histidine residue. We have found that imidazole combines with ferrihemoglobin to form a compound with spectrum differing little from that of ferrihemoglobin hydroxide (alkaline methemoglobin). The data from magnetic titrations at pH 6.86, 8.20, and 10.30 can be interpreted on the assumption of the simple equilibrium

$$HHbIm^+ \rightleftarrows HHb^+ + Im \qquad (1)$$

in which the symbol HHb^+ is used to represent the ferrihemoglobin cation (containing one heme); there is accordingly no evidence from our experiments for heme-heme interactions. The apparent equilibrium constants can be converted into equilibrium constants for Reaction 1 by use of data for the ferrihemoglobin : ferrihemoglobin-hydroxide equilibrium[6] and the imidazole:imidazilium-cation equilibrium (pK 6.95).[7]

The values found at 25°C. for the equilibrium constant $K = [HHb^+][Im]/[HHbIm^+]$ are the following:

pH 6.86	$K = 2.5 \times 10^{-3}$
8.2	2.0×10^{-3}
10.3	0.31×10^{-3}

The probable error in K is about 10% in each case.

The trend in the values of K with change of pH may be due to the presence in imidazole-ferrihemoglobin of an acid group with pK about 9.5, requiring consideration of the equilibrium

$$HHbIm^+ \rightleftarrows HbIm + H^+ \qquad (2)$$

Here we use the symbol HbIm to represent imidazole-ferrihemoglobin after loss of one hydrogen ion per heme. Assuming that this acid group is not effective for ferrihemoglobin itself (within the pH range 6.86 to 10.3), we calculate the ratios 2.3 : 2.2 : 0.31 for K at pH 6.86, 8.2 and 10.3, respectively, in satisfactory agreement with the values reported above.

The equilibrium between imidazole-ferrihemoglobin and its products of dissociation accordingly involves this heme-linked acid group with pK 9.5 as well as the acid group with pK 8.15 which corresponds to the addition of hydroxide ion to ferrihemoglobin ion to form ferrihemoglobin hydroxide. The latter acid group is absent in imidazole-ferrihemoglobin, since presumably imidazole competes with hydroxide ion for coördination with the iron atom.

The molal susceptibility of imidazole-ferrihemoglobin could be evaluated only roughly by extrapolation, the three values 2940, 2180 and 1290 $\times$ 10^{-6} c. g. s. u. being obtained. These correspond to the values 2.66, 2.29

and 1.76 Bohr magnetons for the magnetic moment of the heme group. Although the values are not very precise, there is little doubt that they show the presence of one unpaired electron per heme, with some orbital moment as well as spin moment. (It is possible that the trend of the values is the

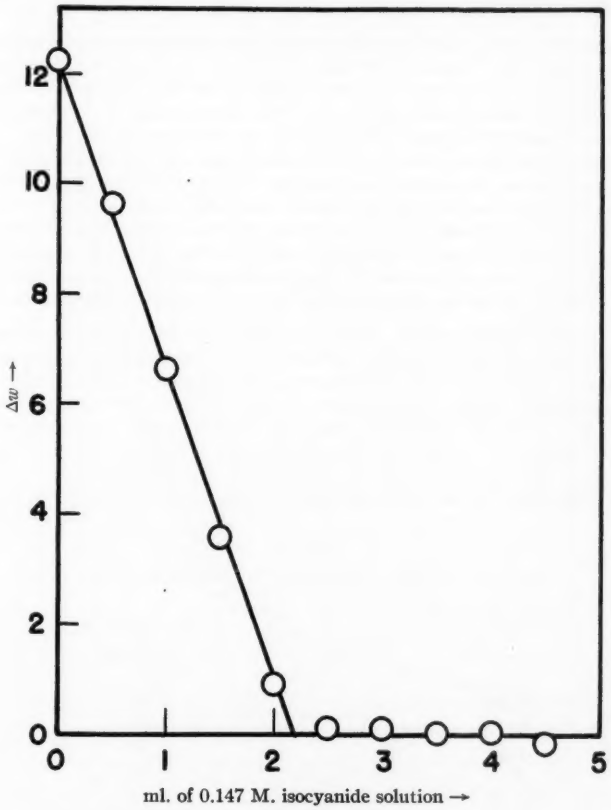


FIGURE 1

Magnetic titration curve for ferrohemoglobin and ethyl isocyanide, showing the formation of the compound ethylisocyanide-ferrohemoglobin.

result of a change in magnetic moment accompanying ionization of the heme-linked acid group.) The bonds to iron in this molecule are hence of the octahedral covalent type, as in ferrihemoglobin cyanide, hydrosulfide, and azide.

We plan to investigate the reactions of ferrihemoglobin with substituted

imidazoles, histidine, and other substances in order to obtain further information about the heme-linked acid group.

We are indebted to Dr. Harrison Davies and Dr. C. D. Coryell for advice and assistance during this work, and to Mr. Ray Clinton for the preparation of ethyl isocyanide.

EXPERIMENTAL PART.—The magnetic measurements were made in the way described in earlier papers.[1, 2, 3] Values of the force exerted by the magnetic field on the two-compartment tube were measured in each experiment for two standard field strengths; the values for the higher field strength were then multiplied by a suitable factor and averaged with those for the lower field strength to give the values of Δw (in mg.) reported in the tables. The values reported are corrected for diamagnetism by use of Δw values of the corresponding carbonmonoxyhemoglobin solutions obtained by reduction with sodium hydrosulfite ($Na_2S_2O_4$) and saturation with carbon monoxide, and are also corrected for dilution by added reagent solutions. The hemoglobin concentrations were calculated from the Δw values with use of data given by Taylor and Coryell[5] for ferrohemoglobin and Coryell, Stitt and Pauling[6] for ferrihemoglobin. Measurements were made at 20°C. for the ferrohemoglobin series and 25°C. for the ferrihemoglobin series.

Ethylisocyanide-ferrohemoglobin.—After a preliminary series of measurements had shown the compound ethylisocyanide-ferrohemoglobin to be largely undissociated under the conditions of the experiment, the following series was made: 20 ml. of a solution of bovine hemoglobin, concentration in heme iron 0.0160 mole/l., were placed in one compartment of a differential susceptibility tube and Δw determined. Successive additions of 0.50-ml. portions of an aqueous solution of ethyl isocyanide, concentration 0.147 mole/l., were then made from a 1-ml. glass syringe, Δw being determined after each addition. The corrected Δw values are given in table 1 and shown in figure 1.

TABLE 1

ADDITION OF ETHYL ISOCYANIDE TO FERROHEMOGLOBIN

TOTAL VOLUME OF ADDED ISOCYANIDE SOLUTION	Δw, CORRECTED
0.00 ml.	12.35 mg.
0.50	9.61
1.00	6.66
1.50	3.59
2.00	0.96
2.50	0.13
3.00	0.11
3.50	0.04
4.00	0.06
4.50	−0.14

Initial volume of ferrohemoglobin solution, 20 ml.
Initial concentration of ferrohemoglobin solution, 0.0160 mole of heme iron/l.
Concentration of ethyl isocyanide solution, 0.147 mole/l.
Temperature, 20°C.

It is seen that the decrease in Δw is linear in the volume of added isocyanide solution, showing the absence of an appreciable amount of uncombined isocyanide. The mole ratio isocyanide/heme iron at the break in the curve is 1.00. The average value relative

to carbonmonoxyhemoglobin of Δw for the last five points, 0.04 mg., differs from zero by less than the probable error of the measurements, showing ethylisocyanide-ferrohemoglobin to contain no unpaired electrons.

Imidazole-ferrihemoglobin.—After preliminary experiments had shown ferrihemoglobin to combine with imidazole, magnetic titrations were made with solutions buffered to

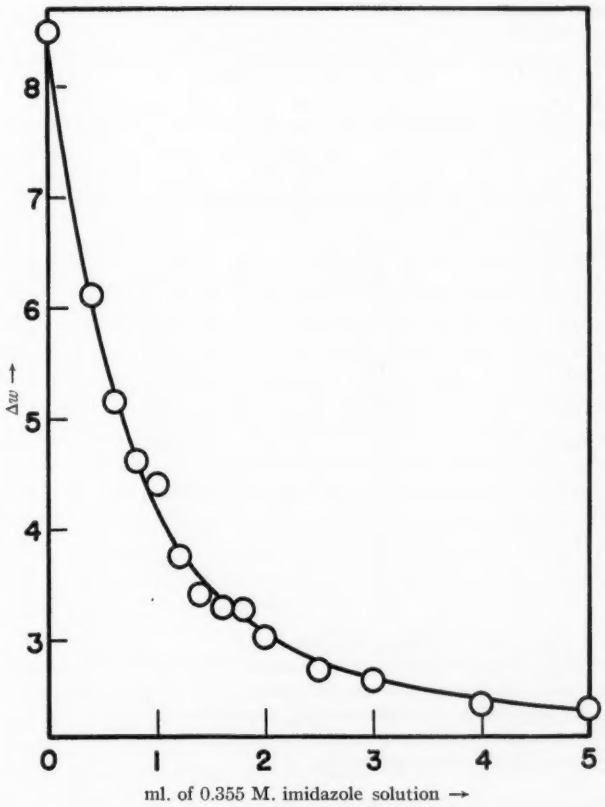


FIGURE 2

Magnetic titration curve for ferrihemoglobin and imidazole at pH 6.86. The full line is the theoretical curve for the value 2.5×10^{-3} for the dissociation constant of the compound.

pH 6.86, 8.20, and 10.30. In each experiment 20 ml. of bovine ferrihemoglobin solution containing phosphate or borate buffer were placed in a differential magnetic tube, Δw was measured, and then successive portions of imidazole solution were added, with measurement of Δw after each addition. In the experiment at pH 6.86 the imidazole solution was brought to this pH before the titration by addition of a small amount of hydrochloric acid. The data for this experiment are given in table 2 and represented in

figure 2. The second column of the table contains the values of Δw corrected for diamagnetism of all constituents and for dilution. In interpreting these values the asymptotic value 1.80 for Δw for the compound imidazole-ferrihemoglobin was selected as leading to no trend in the values of K. The third column contains the concentration of free un-ionized imidazole, calculated from the total added imidazole by correction for the amount combined with ferrihemoglobin and for ionization (44.8% un-ionized at pH 6.86, corresponding to $pK_A = 6.95$[7]). The fourth column contains values of the equilibrium constant K.

Similar experiments were carried out in duplicate at pH 8.2 and pH 10.3, with the following results:

pH 8.2: $\Delta w_{asymptote} = 1.62$ mg.; $K \times 10^3 = 1.7, 2.3$; average 2.0.
pH 10.3: $\Delta w_{asymptote} = 1.00$ mg.; $K \times 10^3 = 0.27, 0.34$; average 0.31.

TABLE 2

ADDITION OF IMIDAZOLE TO FERRIHEMOGLOBIN AT PH 6.86

TOTAL VOLUME OF ADDED IMIDAZOLE SOLUTION	Δw, CORRECTED	(IM)	$K \times 10^3$
0.00 ml.	8.50 mg.	0.00000	...
0.40	6.11	0.00159	2.7
0.60	5.16	0.00253	2.4
0.80	4.61	0.00369	2.5
1.00	4.41	0.00503	3.1
1.20	3.74	0.00611	2.4
1.40	3.42	0.00732	2.2
1.60	3.30	0.00866	2.4
1.80	3.29	0.0101	2.7
2.00	3.03	0.0112	2.4
2.50	2.74	0.0144	2.2
3.00	2.64	0.0175	2.4
4.00	2.42	0.0232	2.3
5.00	2.39	0.0286	2.7
		Average	2.5

Initial volume of ferrihemoglobin solution, 20 ml.
Initial concentration of ferrihemoglobin solution, 0.00974 mole of heme iron/l.
Concentration of imidazole solution, 0.355 mole/l.
Asymptotic value of Δw for imidazole-ferrihemoglobin, 1.80 mg.
χ_{molal} for imidazole-ferrihemoglobin, 2940×10^{-6} c. g. s. u.
Temperature, 25°C. Phosphate buffer.

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CRYSTALLINE ECHINOCHROME AND SPINOCHROME: THEIR FAILURE TO STIMULATE THE RESPIRATION OF EGGS AND OF SPERM OF STRONGYLOCENTROTUS

BY ALBERT TYLER

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Communicated August 28, 1939

The blood cells (elaeocytes) of sea-urchins contain a red pigment to which MacMunn[1] gave the name echinochrome. McClendon[2] found that this pigment is also present in the test and, in the species *Arbacia punctulata*, in the eggs as well. Cannan[3] showed that it could be reversibly oxidized and reduced. Vlès and Vellinger[4] made use of it (called it Arbacine) as a natural pH indicator in the eggs of Arbacia. Stern[5] (referring to unpublished data) considers it to be bound as a chromoprotein in the blood cells.

The present author's interest in echinochrome arose from Friedheim's[6] report that it would cause a 16-fold increase in the respiratory rate of unfertilized sea-urchin (*Strongylocentrotus lividus*) eggs. In view of the magnitude of the reported effect and the fact that the substance is a naturally occurring pigment, it seemed desirable to examine its action on fertilized sea-urchin eggs. This was of especial interest in connection with studies[7] on the energetics of development. However, my attempts to duplicate Friedheim's results on unfertilized sea-urchin eggs have proved unsuccessful. Friedheim did not give in his articles the method of preparation except to state that he extracted the tests of *Sphaerechinus granularis* with sea water. In the present work sea water extracts of *Strongylocentrotus purpuratus* were found to be inactive. The material was then prepared in crystalline form, and the purified echinochrome also was found to have no stimulating effect on the respiration of sea-urchin eggs.

Recently another effect has been attributed to echinochrome: Hartmann, Schartau, Kuhn and Wallenfels[8] have announced that it is the substance causing activation and chemotaxis of the spermatozoa in *Arbacia pustulosa*. They state that the limit of physiological activity lies at a dilution of 1:2,000,000,000. On examination of the action of purified echinochrome on the sperm of *Strongylocentrotus purpuratus* I have found no such action as Hartmann, *et al.*, reported for Arbacia.

The crystallization of echinochrome was first reported by Ball.[9] It was also crystallized by Lederer and Glaser[10] who, though evidently unacquainted with Ball's brief abstract, assigned to it the same empirical formula, $C_{12}H_{10}O_7$. Lederer and Glaser also obtained another pigment which they call spinochrome and which differs from echinochrome in containing one more atom of oxygen. By a similar procedure I was able to obtain

crystalline spinochrome from *Strongylocentrotus purpuratus*. It, too, showed no stimulating effect on the respiration of the eggs and the spermatozoa.

Preparation of Crystalline Echinochrome and Spinochrome.—Ground sea-urchin tests are first extracted with neutral alcohol and then ether. This removes very little of the echinochrome or spinochrome. The test material is then added in successive small portions to 0.5*N* HCl under ether, fresh portions of concentrated HCl being added as the CO_2 evolution ceases. The pigments go almost quantitatively into the ether layer. Shaking the ether solution with aqueous $NaHCO_3$ brings the pigments into the hypo-

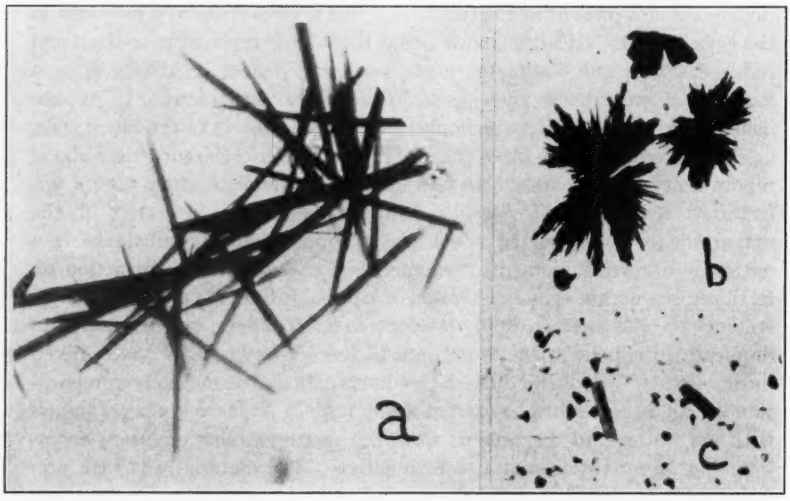


FIGURE 1

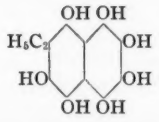
a, Echinochrome crystals × 100; *b*, Spinochrome crystals in rosettes × 50; *c*, Spinochrome crystals × 100.

phase, from which they are returned to fresh ether upon acidification. The dried ether solution is then run through a column of $CaCO_3$. Two main layers appear, an upper bluish violet layer and, below this, a red layer. These two layers remain contiguous even after repeated chromatographic adsorption. The upper layer contains principally the spinochrome, the lower layer the echinochrome. The pigments are "eluted" by dissolving the $CaCO_3$ in acid under ether. The ether is then dried and evaporated to small volume. The addition of petrol ether precipitates the pigment. The precipitate is then fractionally vacuum-sublimed. The sublimate generally deposits in crystalline form, and the corresponding bands of the chromatogram yield principally echinochrome or spinochrome, respectively.

Recrystallization is effected from hot toluene, hot benzene or evaporation of an ether solution containing ligroin.

The crystals of echinochrome (Fig. 1*a*) are bright red elongate needles melting at 220° to 221°C. Spinochrome (Fig. 1*b*, *c*) forms very thin, pale-violet, elongate plates, strongly birefringent and melting at 187° to 188°C.

The investigations of Ball,[9] Lederer[10] and Kuhn[11] show echinochrome to be a substituted naphthoquinone with the following structure in the leuco form:



In spinochrome the location of the extra atom of oxygen has not as yet been determined.

Both echinochrome and spinochrome are only slightly soluble in water or dilute acid. They go readily into solution in alkali. They are very soluble in alcohol, acetone and ether, less soluble in benzene and toluene, insoluble in petroleum ether. The addition of lime water to an aqueous solution of either of these dyes precipitates them almost completely as the calcium salts. It is evidently in this form that the pigments are present in the test of the sea-urchin. A colorimetric comparison of the initial ether extract with standard ether solutions of the purified material shows the pigment concentrations to be roughly 0.5 mg. per gram of dried test.

Action on Respiration of Unfertilized Sea-Urchin Eggs.—Oxygen consumption was measured with the Barcroft-Warburg apparatus. Conical vessels of 15 cc. total volume and with one side arm were employed. The main chamber contained 3.0 cc. of egg suspension and the side arm 1.0 cc. of the solution to be tested. The egg suspensions and solutions were buffered with 0.01 molar glycylglycine.[12]

To test the action of sea-water extracts, tests of *Strongylocentrotus purpuratus* were ground up in sea water. Upon filtration this gives a pale purple opalescent "solution." The addition of this to unfertilized Strongylocentrotus eggs produced no change in their rate of oxygen consumption, as the following figures show:

	A	B
Cu. mm. O_2 consumed per hour before adding solution	28.0	29.5
after	30.0	29.0

The action of purified echinochrome and spinochrome, prepared as described above, was examined both in ordinary sea water and in calcium-free sea water. In ordinary sea water most of the echinochrome or spinochrome precipitates out, presumably as the calcium salt. The addition of

saturated solutions (plus excess solid) of these substances to unfertilized sea-urchin eggs in sea water produced no marked change in the rate of oxygen consumption, as the following figures show:

	ECHINOCHROME		SPINOCHROME	
Cu. mm./hr. before addition	40.0	37.5	37.5	39.5
Cu. mm./hr. after addition	39.0	38.5	40.0	36.5

In calcium-free sea water at pH 8, fairly concentrated solutions of echinochrome and of spinochrome could be prepared. The action of these solutions was tested over a range of concentrations since there might be an optimum concentration above which no stimulation is obtained, as is the case with the substituted phenols.[13,14] Echinochrome and spinochrome were added to unfertilized Strongylocentrotus eggs in calcium-free sea water to give concentrations ranging from 0.01 mg./cc. to 1.0 mg./cc. To check on the stimulability of the respiratory rate of the eggs in calcium-free sea water the action of pyocyanine was also examined. The following table gives a sample of the data obtained.

	ECHINOCHROME			SPINOCHROME		PYOCYANINE	
Mg./cc.	0.01	0.1	1.0	0.01	0.1	1.0	0.5
Cu. mm./hr. before	36.5	37.5	35.5	36.0	37.0	39.0	38.5
Cu. mm./hr. after	37.0	41.5	40.5	40.5	34.0	35.0	307.5

Although pyocyanine gave an eight-fold rise in the respiratory rate of the eggs in calcium-free sea water, echinochrome and spinochrome failed to show any definite effect.

It is somewhat questionable whether the dyes penetrated the eggs in these solutions. While in the solution, it is difficult to tell if the eggs are stained. Upon removal and washing, the eggs are unstained, which may mean rapid diffusion or failure to penetrate initially. If the solutions are acidified to below pH 3, the eggs are colored red by the dyes, and the color remains upon returning the eggs to ordinary sea water. But the eggs are killed by such treatment.

In the eggs of sea-urchins of the genus Arbacia, the echinochrome may very well be present in the form of the chromoprotein, as Stern has indicated is the case in the blood. Possibly the eggs of Strongylocentrotus and other forms which do not contain echinochrome also lack the protein capable of combining with it.

Friedheim[6] reported obtaining stimulation of the respiratory rate of rabbit red blood cells as well as of eggs of *S. lividus*. The present negative results with eggs of *S. purpuratus* are, then, evidently not due to species differences. It would appear that in the crude extracts of Sphaerechinus tests employed by Friedheim something other than echinochrome was responsible for the stimulating effect.

Action on the Respiration of Sea-urchin Spermatozoa.—When sea-urchin spermatozoa are placed in calcium free sea water they are rapidly immobil-

ized. For the respiration measurements, the spermatozoa were therefore suspended in ordinary sea water (buffered) and the solutions of the dyes in isotonic NaCl (buffered) were added. As a control, isotonic NaCl alone was added, and, as check on the stimulability of the respiration, an egg water in isotonic NaCl was employed. The addition of the solutions of the dye to sea water results in their precipitation as the calcium salts. With high concentrations of dye practically all of the calcium would be removed from the sea water. With low concentrations, on the other hand, very little dye would remain in solution.

Two sets of experiments were run, one with low and one with high concentrations of the dye. With low concentrations (10 γ/cc.) of echinochrome and spinochrome, there was no noticeable effect on the rate of oxygen uptake of the spermatozoa. With high concentrations of the dyes (10 mg./cc.), the rate is definitely inhibited as the data in the following table show.

		ECHINOCHROME		SPINOCHROME		EGG WATER	ISOTONIC NACL (CONTROL)
Cu. mm. O_2 before addition	0 to 1/2 hr.	103	102	97	98	98	99
	1/2 to 1 hr.	81	83	73	74	77	77
	1 to 1 1/2 hr.	59	69	59	57	56	57
Cu. mm. O_2 after addition	1 1/2 to 2 hr.	8	11	12	12	57	41
	2 to 2 1/2 hr.	7	8	3	7	37	35
	2 1/2 to 3 hr.	13	11	1	2	26	21
	3 to 3 1/2 hr.	12	11	4	3	22	19

The activity of spermatozoa decreases with age of the suspensions, as has long been known. Along with that, the respiratory rate falls off as the above figures (control column and first three readings of the others) show. The addition of egg water causes an increase of about 40 per cent in the respiratory rate for the first half hour and practically no increase for the subsequent readings. The concentration of egg water employed was such as to give an agglutination reaction lasting 15 minutes. The sharp decrease in respiratory rate following the addition of echinochrome and spinochrome is evidently due to the removal of calcium. Most of the dye was observed to precipitate out upon addition to the sea water sperm suspension in the vessels. This precipitate is very likely the calcium salt, since as mentioned before echinochrome and spinochrome form insoluble salts with calcium. With magnesium, which is present in higher concentrations than calcium in sea water, no insoluble salt forms. The 3.0 cc. of sea water contains roughly 3×10^{-5} moles of Ca^{++}. In the 1.0 cc. of echinochrome solution added there are 3.8×10^{-5} moles of the dye, and in the spinochrome solution, 3.5×10^{-5} moles. Since one mole of echinochrome or spinochrome might combine with 2 1/2 moles of calcium, most of the calcium would be removed from the solution. Examination of the

suspensions after the respiration measurements showed the spermatozoa to be almost completely inactive in the vessels with the dyes. This inactivation occurs in 2 to 3 minutes in concentrations of the dyes employed here.

The difference between the present results and those reported by Hartmann, *et al.*,[8] may possibly be due to the fact that Arbacia eggs contain echinochrome whereas Strongylocentrotus eggs do not. It should be noted, however, that Strongylocentrotus egg water increases the activity of spermatozoa of the same species while echinochrome and spinochrome show no such action. The activating agent in the egg water of this sea-urchin as well as in the keyhole limpet is rather intimately associated with the sperm agglutinin which recent evidence[15] shows to be protein. Partial purification of the latter has not as yet removed the activating agent.

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THE FINITE STIELTJES MOMENTUM PROBLEM

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I. *Introduction.*—In view of the frequent application of Ritz's method to the solution of eigenvalue problems $L\Psi = \lambda\Psi$ increasing attention has been paid to the study of types of function sets which facilitate the performance of variational calculations. In several numerical calculations it was noted[1, 2, 3] that the iterated function set $L^i\psi^4$ lead to relatively simple matrix elements of the form $L_n = (\psi, L^n\psi)$ and that the convergence of the

method was unexpectedly good. But it was realized only recently that the method commends itself also in other respects, viz., that the information contained in the momenta L_n extends considerably beyond the results of the secular equation.[5] D. H. Weinstein[6] was the first to discover a result pointing in a new direction. He found that under suitable conditions the expression $L_1 - (L_2 - L_1^2)^{1/2}$ defined a lower bound for the lowest eigenvalue of L. The form of the expression suggested that it is but a very special case of a more general set of relations, and the slight extensions of Weinstein's theorem obtained by MacDonald,[7] Stevenson[8] and Horvay[9] confirmed this point of view. In this note we present the general relations. They were found to be intimately connected with the theory of the finite Stieltjes momentum problem. We formulate the latter as follows: given a *finite* number of momenta $L_0, L_1, \ldots, L_n$, find *all* the information which they imply about the spectrum of L.

II. *The Momentum Problem.*—Let L be a self-adjoint operator operating on a certain class of admissible functions.[10] The part of the spectrum of L which is associated with states not accurately orthogonal to a function ψ will be referred to as the *simplified spectrum* of L with respect to ψ. In what follows we shall assume that n point-eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ constitute the part of the simplified spectrum of L which is $\leqslant \lambda_n$.

Let ψ be a normalized admissible function such that $L^1\psi, \ldots, L^{n-1}\psi$ also are admissible. The momenta $L_0 = 1, L_1, \ldots, L_n$ of L, obtained in terms of ψ, are supposed to be known. Let x_m^i be the elementary symmetric forms of order i in the m variables $x_1, x_2, \ldots, x_m$, i.e., $x_m^0 = 1, x_m^1 = x_1 + x_2 + \ldots + x_m, \ldots, x_m^m = x_1 x_2 \ldots x_m$. Further set $L_{m+j,\,m}(x_1, \ldots, x_m) = L_{m+j} - x_m^1 L_{m+j-1} + x_m^2 L_{m+j-2} - \ldots + (-1)^m x_m^m L_j$, and denote

$$ {}^r\Omega_m(x_1, \ldots, x_m) = \begin{vmatrix} L_{m,m} & L_{m+1,m} & \cdot & \cdot & \cdot & L_{m+r,m} \\ L_{m+1,m} & L_{m+2,m} & \cdot & \cdot & \cdot & L_{m+r+1,m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ L_{m+r,m} & L_{m+r+1,m} & \cdot & \cdot & \cdot & L_{m+2r,m} \end{vmatrix}. $$

Then the following *fundamental theorem* holds:

$$ {}^r\Omega_m(\lambda_1, \ldots, \lambda_m) > 0 \qquad (0 \leqslant r,\ 0 \leqslant m,\ m + 2r \leqslant n), \quad (F.T.) $$

provided that $\lambda_1, \ldots, \lambda_n$ do not exhaust the entire simplified spectrum. (If they do, then the relation ${}^0\Omega_0 > 0$ must be replaced by ${}^0\Omega_n = 0$ in $(F.T.)$.*) The relations ${}^r\Omega_0 > 0$ do not restrict the eigenvalues λ_i, they express the well-known conditions for the solution of the proper Stieltjes problem. The inequalities ${}^r\Omega_1(x_1) > 0$ restrict λ_1 to values smaller than the lowest root of the secular equations, ${}^r\Omega_1(x) = 0$, which express the familiar minimum conditions of the variational method[5] applied to functions $\sum_0^r A_i L^i\psi$. The relations $m > 1$ have not been considered heretofore.

Proof of (*F.T.*).—It follows from the theory of self-adjoint operators[11] that if the polynomial $f(\lambda) = \sum_0^m c_i\lambda^i$ ($c_m \neq 0$, $m \leqslant n$) is non-negative over the simplified spectrum of L ($f(\lambda_1) \geqslant 0, f(\lambda_2) \geqslant 0, \ldots$), then the "average value" of $f(\lambda)$, $<f>$, is positive: $<f> = (\psi, f(L)\psi) = \sum_0^m c_i L_i > 0$. Thus $<(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_m)> = L_m - \lambda_m^1 L_{m-1} + \lambda_m^2 L_{m-2} - \ldots + (-1)^m \lambda_m^m L_0 = L_{m,m}(\lambda_1, \ldots, \lambda_m) > 0$. More generally $<(\sum_0^r \gamma_i \lambda^i)^2(\lambda - \lambda_1) \ldots (\lambda - \lambda_m)> = \sum_{i,j} \gamma_i \gamma_j L_{m+i+j,m}(\lambda_1, \ldots, \lambda_m) > 0$. The determinant of this positive definite quadratic form, $|L_{m+i+j,m}| = {}^r\Omega_m(\lambda_1, \ldots, \lambda_m)$, is also positive, q. e. d.

The inequalities (*F.T.*) prescribe conditions which are satisfied by the eigenvalues of L. It is as yet an open question whether this set of inequalities is complete,** i.e., whether, conversely, to each set of numbers $\lambda_1, \ldots, \lambda_n$ and $L_1, \ldots, L_n$ satisfying (*F.T.*) there exist an operator L and a function ψ such that $\lambda_1, \ldots, \lambda_n$ constitute the part $\leqslant \lambda_n$ of the simplified spectrum, $L_1, \ldots, L_n$ the first n momenta of L in terms of ψ.

III. *Discussion of Special Cases.—Weinstein's assumption and assumptions of a similar type.*

In this section we shall consider in some detail the simplest conditions ${}^r\Omega_{s+1} > 0$ ($r = 0,1$; $s = 0,1$). They give insight as to the relations which prevail in the general case.

(1) ${}^0\Omega_1(x_1) = L_1 - x_1$ vanishes for $x_1 = {}^0\lambda_1 = L_1$. The condition ${}^0\Omega_1(\lambda_1) > 0$ leads to the well-known result $\lambda_1 < {}^0\lambda_1$. The "admitted region" of points $x_1 < {}^0\lambda_1$ is denoted by ${}^0U_1[x_1]$. ${}^0U_1$ is bounded by $x_1 = {}^0\lambda_1$ on the right, extends indefinitely on the left. It contains λ_1.

(2) ${}^0\Omega_2(x_1, x_2) = L_2 - (x_1 + x_2)L_1 + x_1x_2$ vanishes along the hyperbola ${}^0\Omega_2 = 0$ in the $[x_1, x_2]$ plane. $x_1 = {}^0\lambda_1$, $x_2 = {}^0\lambda_1$ are the asymptotes. The conditions ${}^0\Omega_2(\lambda_1, \lambda_2) > 0, \lambda_2 > \lambda_1, {}^0\Omega_1(\lambda_1) > 0$ restrict the location of the point (λ_1, λ_2) to the admitted region ${}^0U_2[x_1, x_2]$ which is bounded from above by the hyperbola, from below by the line $x_2 = x_1$, and from the right by the line $x_1 = {}^0\lambda_1$. Between the lines $x_2 = x_1$, $x_2 = {}^0\lambda_1$ it extends indefinitely on the left. Often, however, one may contend that the point (λ_1, λ_2) is not likely to be outside of some conveniently drawn boundary line. (For example the experimental knowledge of the spectrum of an operator L' not much different from L may suggest appropriate limitations within ${}^0U_2$.) Several such limitations have been considered. (*a*) Weinstein's condition,[6] L_1 is closer to λ_1 than to λ_2, limits the admitted region to the right-side of the line $x_2 - L_1 = -(x_1 - L_1)$. The abscissa of the point farthest to the left gives $L_1 - (L_2 - L_1^2)^{1/2}$ as a lower bound for λ_1. (*b*) The assumption of a lower bound p for $P \equiv -(\lambda_2 - L_1)/(\lambda_1 - L_1)$ corresponds to a slight weakening of Weinstein's restriction[9] and implies that the point (λ_1, λ_2) is above the line $x_2 - L_1 = -p(x_1 - L_1)$. This lines goes through the point (L_1, L_1) and intersects the hyperbola ${}^0\Omega_2 = 0$ for $p > 0$ in the point $(L_1 - p^{-1/2}(L_2 - L_1^2)^{1/2}, L_1 + p^{1/2}(L_2 - L^2)^{1/2})$ which is to the left of $x_1 = \lambda_1$. (*c*) The

restriction of the point (λ_1, λ_2) to the region above a line $x_1 + x_2 = \text{const.}$ drawn slightly below the experimental point (λ_1', λ_2') leads to Stevenson's formula.[8] (*d*) If λ_1 is assumed to be known exactly, then the intersection of the ordinate $x_1 = \lambda_1$ with ${}^0\Omega_2 = 0$ yields an upper bound for λ_2.

Some of the formulae become simpler if one shifts the spectrum of L to have L_1 as the origin. The operator $D = L - L_1$, with the simplified spectrum $\delta_n = \lambda_n - L_1$ and momenta $D_n = \sum_0^n \binom{n}{i}(-L_1)^i L_{n-i}$ gives $D_0 = 1$, $D_1 = 0$. Denoting $z_n = x_n - L_1$, one can write ${}^0\Omega_1(z_1) = -z_1$, ${}^0\Omega_2(z_1, z_2) = -{}^0\Omega_1(z_1)[z_2 + D_2/z_1]$. We shall adhere to this notation in the examples which follow.

(3) ${}^1\Omega_1(z_1) = {}^1\Omega_0[z_1^2 - z_1 D_3/D_2 - D_2] = 0$ is the secular equation obtained in terms of the variational function $\psi + aD\psi$. The two roots of the equation, $z_1 = {}^1\delta_1 = \frac{1}{2}D_3/D_2 - \left\{(\frac{1}{2}D_3/D_2)^2 + D_2\right\}^{1/2}$ and $z_1 = {}^1\delta_2 = \frac{1}{2}D_3/D_2 + \left\{(\frac{1}{2}D_3/D_2)^2 + D_2\right\}^{1/2}$ are separated by the root ${}^0\delta_1 = 0$ of ${}^0\Omega_1(z_1) = 0$. The conditions ${}^0\Omega_1(\delta_1) > 0$ and ${}^1\Omega_1(\delta_1) > 0$ lead to the well-known result $\delta_1 < {}^1\delta_1$. ${}^1U_1[z_1]$, the admitted region of points $z_1 < {}^1\delta_1$, is bounded by $z_1 = {}^1\delta_1$ on the right, extends indefinitely on the left. It contains δ_1. From the conditions ${}^0\Omega_2 > 0$ and ${}^1\Omega_1 > 0$; it follows that ${}^1\delta_1{}^1\delta_2 = -D_2 < \delta_1\delta_2$ and ${}^1\delta_1 + {}^1\delta_2 = D_3/D_2 > \delta_1 + \delta_2$. One sees that $D_3 < 0$ implies $D_1 = 0 > \frac{1}{2}(\delta_1 + \delta_2)$. The knowledge of ${}^1\delta_1$ yields an upper bound for $P = -\delta_2/\delta_1$, namely, $P < D_2/\delta_1^2 < D_2/{}^1\delta_1^2$. The value of P is of interest because it tells us how D_1, the zeroth approximation of δ_1, divides the interval $\{\delta_1, \delta_2\}$. If D_1 is inside this interval then the ratio P is positive and is close to ∞ when D_1 is close to δ_1 (i.e., when ψ is a good approximation to the state associated with δ_1).

(4) ${}^1\Omega_2(z_1, z_2) = {}^1\Omega_1(z_1)[z_2^2 - z_2(D_3/D_2)\{{}^1Z_1(z_1)/{}^1\Omega_1(z_1)\} - D_2 + {}^2\Omega_0/{}^1\Omega_1(z_1)]$ where ${}^1Z_1(z) = {}^1\Omega_0[z^2 - z(D_4 - D_2^2)/D_3 - D_2]$. ${}^1\Omega_2$ vanishes in the $[z_1, z_2]$ plane along two branches which are separated by the hyperbola ${}^0\Omega_2 = 0$. $z_1, z_2 = {}^1\delta_1, {}^1\delta_2$ are the asymptotes. The conditions $\delta_2 > \delta_1$, ${}^0\Omega_1 > 0$, ${}^1\Omega_1 > 0$, ${}^0\Omega_2 > 0$, ${}^1\Omega_2 > 0$ restrict the location of the point (δ_1, δ_2) to the admitted region ${}^1U_2[z_1, z_2]$ which is bounded from above by the lower branch of ${}^1\Omega_2 = 0$, from below by the line $z_2 = z_1$ and from the right by the line $z_1 = {}^1\delta_1$. Between the lines $z_2 = z_1$, $z_2 = {}^1\delta_1$ it extends indefinitely on the left. At the point $({}^1\delta_1, {}^1\delta_2)$, ${}^0\Omega_2 = 0$ touches the lower branch of ${}^1\Omega_2 = 0$ from above, and at the point $({}^2\delta_1, {}^2\delta_2)$ this branch in turn touches the lowest branch of ${}^2\Omega_2 = 0$ from above. ${}^2\delta_1 < {}^2\delta_2 < {}^2\delta_3$ denote the three roots of the cubic equation ${}^2\Omega_1(z) = 0$. The rather weak restriction $\delta_2 > 0$ (i.e., the zeroth approximation L_1 is already below λ_2) leads to the lower bound $\frac{1}{2}D_3/D_2 - \left\{(\frac{1}{2}D_3/D_2)^2 + D_2 + {}^2\Omega_0/D_2^2\right\}^{1/2} < \delta_1$.

The most important properties of the functions ${}^r\Omega_{s+1}(x_1, \ldots, x_{s+1})$ can now be stated in general by analogy. ${}^r\Omega_{s+1}$ vanishes along $r + 1$ sheets separated by the r sheets, all different, of ${}^{r-1}\Omega_{s+1} = 0$. The lowest sheets of ${}^r\Omega_{\sigma+1} = 0$, ${}^{r-1}\Omega_{\sigma+1} = 0, \ldots, {}^{r-\sigma}\Omega_{\sigma+1} = 0$ (omit expressions with negative indices) touch at the point $({}^r\lambda_1, \ldots, {}^r\lambda_{\sigma+1})$ constructed from the $\sigma + 1$

lowest roots ${}^{r}\lambda_1 < {}^{r}\lambda_2 < \ldots < {}^{r}\lambda_{\sigma+1}$ of the secular equation ${}^{r}\Omega_1(x) = 0$. The first σ of these sheets touch along a common line connecting the points $({}^{r}\lambda_1, \ldots, {}^{r}\lambda_{\sigma+1})$ and $({}^{r+1}\lambda_1, \ldots, {}^{r+1}\lambda_{\sigma+1})$, the first $\sigma - 1$ of these along a common surface, etc. The point $(\lambda_1, \ldots, \lambda_{s+1})$ will be located in the *admitted region* ${}^{r}U_{s+1}[x_1, \ldots, x_{s+1}]$, which is defined as the set of points $x_1 < x_2 < \ldots < x_{s+1}$ satisfying the conditions ${}^{\rho}\Omega_{\sigma+1}(x_1, \ldots, x_{\sigma+1}) > 0$ $(\sigma + 2\rho \leqq s + 2r, \sigma \leqslant s)$. The rôle of assumptions of Weinstein's type is now apparent. One eliminates a portion of the admitted region by some surface and restricts the point $(\lambda_1, \ldots, \lambda_{s+1})$ to the remainder of the region. Thus it is possible to obtain lower bounds.[12]

IV. *Discussion of the Admitted Regions.*—The admitted regions ${}^{r}U_s$, defined by the conditions $x_1 < x_2 < \ldots < x_s$ and ${}^{\rho}\Omega_\sigma > 0$ $(2\rho + \sigma \leqslant 2r + s, \sigma \leqslant s)$, are of fundamental importance in the theory because they serve to restrict the point $(\lambda_1, \ldots, \lambda_s)$ to a certain portion of the $[x_1, \ldots, x_s]$ space. The functions ${}^{r}\Omega_s$ owe their significance to the fact that the regions ${}^{r}U_s$ are characterized in terms of them. Let $(x_1, \ldots, x_s)$ be a fixed point in the intersection ${}^{r}U_s \cdot {}^{r+1}U_{s-1}$, i.e., let $(x_1, \ldots, x_s)$ be in ${}^{r}U_s$, $(x_1, \ldots, x_{s-1})$ in ${}^{r+1}U_{s-1}$. Then the theorem holds:

THEOREM $({}^{r}_{s})_{\mathrm{I}}$.—The equation ${}^{r}\Omega_{s+1}(x_1, \ldots, x_s, x) = 0$ has $r + 1$ real different roots $x = {}^{r}_{s}\Lambda_1, {}^{r}_{s}\Lambda_2, \ldots, {}^{r}_{s}\Lambda_{r+1}$ separated by $x_s, {}^{r-1}_{s}\Lambda_1, \ldots, {}^{r-1}_{s}\Lambda_r$: $x_s < {}^{r}_{s}\Lambda_1 < {}^{r-1}_{s}\Lambda_1 < {}^{r}_{s}\Lambda_2 < \ldots < {}^{r-1}_{s}\Lambda_r < {}^{r}_{s}\Lambda_{r+1}$. We shall postpone the proof of this theorem until the end of the section and shall first discuss its various consequences $(({}^{r}_{s})_{\mathrm{II, III}, c, 1, 2, 3})$ under the assumption that the theorems $({}^{k}_{l})_{\mathrm{I}}$ as well as $({}^{k}_{l})_{\mathrm{II}, 1, 2}$ already hold for $k + l \leqslant r + s$, $k \leqslant r - 1$. Since $({}^{r}_{s})_{\mathrm{I, II, III}, c, 1, 2, 3}$ obviously hold for $r = 0$, s arbitrary, it will be sufficient to assume $r \geqslant 1$ in the proofs which follow.

COROLLARY $({}^{r}_{s})_c$. ${}^{r}\Omega_{s+1} > 0$ for $x_{s+1} < {}^{r}_{s}\Lambda_1$.

PROOF. ${}^{r}\Omega_{s+1}$ is a polynomial of order $r + 1$ in x_{s+1}, because the factor ${}^{r}\Omega_s$ in the term of highest order, ${}^{r}\Omega_s \cdot (-x_{s+1})^{r+1}$, is positive for $(x_1, \ldots, x_s)$ $\epsilon {}^{r}U_s$. This term predominates for large values of $|x_{s+1}|$ and makes ${}^{r}\Omega_{s+1}$ positive in the interval $-\infty < x_{s+1} < {}^{r}_{s}\Lambda_1$.

THEOREM $({}^{r}_{s})_{\mathrm{II}}$. ${}^{r}U_{s+1}$ is that set of points $(x_1, \ldots, x_s, x_{s+1})$ for which $(x_1, \ldots, x_s) \epsilon {}^{r}U_s \cdot {}^{r+1}U_{s-1}$ and $x_s < x_{s+1} < {}^{r}_{s}\Lambda_1$.

We observe that ${}^{r}U_{s+1}$ can evidently be characterized as the set of points $(x_1, \ldots, x_{s+1})$ which satisfy ${}^{r}\Omega_{s+1} > 0$ and are in ${}^{r-1}U_{s+1} \cdot {}^{r}U_s \cdot {}^{r+1}U_{s-1}$. In view of $({}^{r-1}_{s})_{\mathrm{II}}$, the condition $(x_1, \ldots, x_{s+1}) \epsilon {}^{r-1}U_{s+1}$ can here be replaced by the condition $x_s < x_{s+1} < {}^{r-1}_{s}\Lambda_1$. According to $({}^{r}_{s})_{\mathrm{I}}$, $({}^{r}_{s})_c$ the latter condition together with ${}^{r}\Omega_{s+1} > 0$ can be replaced by $x_s < x_{s+1} < {}^{r}_{s}\Lambda_1$. This establishes the theorem.

${}^{r}_{s}\Lambda_1$, the lowest solution of ${}^{r}\Omega_{s+1}(x_1, \ldots, x_s, x) = 0$ in accordance with theorem $({}^{r}_{s})_{\mathrm{I}}$, is called the first adjoint of $x_1, \ldots, x_s$ and will be denoted by ${}^{r}\Lambda_{s+1}$. The point $(x_1, \ldots, x_s, {}^{r}\Lambda_{s+1})$ is in ${}^{r-1}U_{s+1} \cdot {}^{r}U_s$. The lowest solution of ${}^{r-1}\Omega_{s+2}(x_1, \ldots, x_s, {}^{r}\Lambda_{s+1}, x) = 0$ defined according to theorem

$\binom{r-1}{s+1}_I$, is called the second adjoint of $x_1, \ldots, x_s$ and will be denoted by ${}^{r-1}\Lambda_{s+2}$. The point $(x_1, \ldots, x_s, {}^r\Lambda_{s+1}, {}^{r-1}\Lambda_{s+2})$ is in ${}^{r-2}U_{s+2} \cdot {}^{r-1}U_{s+1}$. Continuing in this way, we obtain the adjoint set ${}^r\Lambda_{s+1}, {}^{r-1}\Lambda_{s+2}, \ldots, {}^0\Lambda_{s+r+1}$. As will be shown below, the adjoint set and the set ${}^r_s\Lambda_1, {}^r_s\Lambda_2, \ldots, {}^r_s\Lambda_{r+1}$ are identical. Denote by y and member of the adjoint set, by Λ^i_r the ith symmetric form of the remaining r adjoint quantities, by Λ^i_{r+1} the ith symmetric form of all $r+1$ adjoint quantities. Then we have:

LEMMA $\binom{r}{s}$. For the set $x_1, \ldots, x_s, {}^r\Lambda_{s+1}, \ldots, {}^0\Lambda_{s+r+1}$, the following formulae hold:

$$L_{s+r+1+j,\, s+r+1} = 0 \qquad (j = 0, 1, \ldots, r) \qquad \binom{r}{s}_1$$

$${}^r\Omega_s \cdot L_{s+2r+2,\, s+r+1} = {}^{r+1}\Omega_s \qquad \binom{r}{s}_2$$

$${}^{r+1}\Omega_{s+1}(x_1, \ldots, x_s, y)[{}^r\Omega_s]^2 + {}^{r-1}\Omega_{s+1}(x_1, \ldots, x_s, y)[{}^{r+1}\Omega_s]^2 = 0 \qquad \binom{r}{s}_3$$
$$({}^{-1}\Omega_{s+1} = 0).$$

Before proving these relations, we shall draw a few conclusions. Upon eliminating the r adjoint quantities $\neq y$ from equations $\binom{r}{s}_1$ we obtain ${}^r\Omega_{s+1}(x_1, \ldots, x_s, y) = 0$. This establishes the above-mentioned identity between the adjoint set and the roots of the equation ${}^r\Omega_{s+1}(x_1, \ldots, x_s, x) = 0$. If we eliminate from the equations $\binom{r}{s}_1$ only $s + r - \sigma$ adjoint quantities and denote those not eliminated by $y_{s+1} < \ldots < y_{\sigma+1}$ we find that the determinants ${}^\rho\Omega_{\sigma+1}$ $(\rho = s + r - \sigma, \ldots, r)$ vanish. An immediate consequence is

THEOREM $\binom{r}{s}_{III}$. Let $y_{s+1} < \ldots < y_{\sigma+1}$ be a subset in ${}^r_s\Lambda_1, \ldots, {}^r_s\Lambda_{r+1}$, then ${}^\rho\Omega_{\sigma+1}(x_1, \ldots, x_s, y_{s+1}, \ldots, y_{\sigma+1}) = 0$ $(\rho \leqslant r, s \leqslant \sigma \leqslant s + r \leqslant \sigma + \rho)$. The special cases $s = 0, 1, 2, \ldots$ lead to points, branches, sheets, ... in which the functions ${}^\rho\Omega_{\sigma+1}$ vanish. We already met such relations in section III, where we discussed explicitly the cases $r = 0, 1$; $s = 0, 1$. In that section we also gave a generalization of the observed conditions substantiated now by theorems $\binom{r}{0}_{I, III}$ on identifying ${}^r\lambda_i$ with ${}^r_0\Lambda_i$.

We now introduce the proof of lemmas $\binom{r}{s}_{1, 2, 3}$. Consider the point $(x_1, \ldots, x_s, {}^r\Lambda_{s+1})$. From $\binom{r-1}{s+1}_1$ we obtain $L_{s+r+1+j,\, s+r+1} = 0$ for $j = 0, 1, \ldots, r-1$ while $L_{s+2r+1,\, s+r+1} = 0$ results from $\binom{r-1}{s+1}_2$ observing ${}^r\Omega_{s+1}(x_1, \ldots, x_s, {}^r\Lambda_{s+1}) = 0$ and ${}^{r-1}\Omega_{s+1}(x_1, \ldots, x_s, {}^r\Lambda_{s+1}) > 0$. This proves $\binom{r}{s}_1$. In order to establish lemma $\binom{r}{s}_2$ we observe that the value of the determinant ${}^{r+1}\Omega_s$ remains unchanged if we replace the last column $L_{s+r+1+j,\, s}$ $(j = 0, 1, \ldots, r+1)$ by a linear combination of the columns, viz., $L_{s+r+1+j,\, s+r+1} = L_{s+r+1+j,\, s} - \Lambda^1_{r+1}L_{s+r+j,\, s} + \Lambda^2_{r+1}L_{s+r-1+j,\, s} - \ldots + (-1)^{r+1}\Lambda^{r+1}_{r+1}L_{s+j,\, s}$. All elements except $L_{s+2r+2,\, s+r+1}$ vanish in this column, and the determinant reduces to the form given in $\binom{r}{s}_2$. If in a similar way we replace the last column in ${}^{r+1}\Omega_{s+1}(x_1, \ldots, x_s, y)$ by $L_{s+r+2+j,\, s+r+1}$, the preceding column by $L_{s+r+1+j,\, s+r+1}$, employing the symmetric functions Λ^i_r, we find ${}^{r+1}\Omega_{s+1} = -{}^{r-1}\Omega_{s+1} \cdot [L_{s+2r+2,\, s+r+1}]^2$. Application of $\binom{r}{s}_2$ gives $\binom{r}{s}_3$.

We shall now derive $({}^{r+1}_{s})_{\mathrm{I}}$. Let $(x_1, \ldots, x_s)$ be in ${}^{r+1}U_s$. By theorem $({}^{r}_{s})_{\mathrm{I}}$ the sign of ${}^{r-1}\Omega_{s+1}(x_1, \ldots, x_s, x)$ is alternatively positive and negative as x runs through the values ${}^{r}_{s}\Lambda_1, \ldots, {}^{r}_{s}\Lambda_{r+1}$. By $({}^{r}_{s})_3$, ${}^{r+1}\Omega_{s+1}(x_1, \ldots, x_s, x)$ takes on the opposite sign at these values; furthermore it has a root below ${}^{r}_{s}\Lambda_1$ and also has a root above ${}^{r}_{s}\Lambda_{r+1}$, because the dominating term ${}^{r+1}\Omega_s \cdot (-x)^{r+2}$ makes the expression positive for large values of $-x$ and takes on a sign opposite to the sign of ${}^{r+1}\Omega_{s+1}(x_1, \ldots, x_s, {}^{r}_{s}\Lambda_{r+1})$ for large values of $+x$. Consequently, the equation ${}^{r+1}\Omega_{s+1}(x_1, \ldots, x_s, x) = 0$ has $r+2$ roots ${}^{r+1}_{s}\Lambda_1, \ldots, {}^{r+1}_{s}\Lambda_{r+2}$ separated by ${}^{r}_{s}\Lambda_1, \ldots, {}^{r}_{s}\Lambda_{r+1}$. It remains to be shown that $x_s < {}^{r+1}_{s}\Lambda_1$. Introduce the condition $(x_1, \ldots, x_{s-1})\epsilon^{r+2}U_{s-1}$. Since ${}^{\rho}\Omega_{s-1} > 0$ $(\rho = 0, 1 \ldots, r+2)$ the quadratic form $\sum_{i,j=0}^{r+2} a_i a_j L_{s-1+i+j,\, s-1}$ is positive definite and the form $\sum_{i,j=0}^{r+1} b_i b_j L_{s+1+i+j,\, s+1}(x_1, \ldots, x_{s-1}, x, x) = \sum_{i,j=0}^{r+2}(b_{i-1} - xb_i)(b_{j-1} - xb_j)L_{s-1+i+j,\, s-1}$ $(b_{-1} = b_{r+2} = 0)$ is positive definite *a fortiori*. Hence the determinant ${}^{r+1}\Omega_{s+1}$ of this form is positive. For $x = x_s$ this results in ${}^{r+1}\Omega_{s+1}(x_1, \ldots, x_{s-1}, x_s, x_s) > 0$ which in comparison with ${}^{r+1}\Omega_{s+1}(x_1, \ldots, x_s, {}^{r}_{s}\Lambda_1) < 0$ indicates that the smallest solution of ${}^{r+1}\Omega_{s+1} = 0$ lies between x_s and ${}^{r}_{s}\Lambda_1$. Thus we have shown that $({}^{r+1}_{s})_{\mathrm{I}}$ is a consequence of $({}^{r}_{s})_{\mathrm{I}}$ and $({}^{k}_{l})_{\mathrm{I}}$ $(k + l \leqslant r + s,\ k \leqslant r)$. Since theorem $({}^{0}_{s})_{\mathrm{I}}$ obviously holds for all values of s, theorem $({}^{r}_{s})_{\mathrm{I}}$ is established by induction.

* We shall exclude this case in what follows.

** Addition in the proof: The set is not complete; the higher eigenvalues are restricted by further conditions.

[1] H. R. Hasse, *Proc. Cambridge Phil. Soc.*, **26**, 542 (1930).

[2] E. Feenberg and J. K. Knipp, *Phys. Rev.*, **48**, 906 (1935).

[3] D. T. Warren and H. Margenau, *Ibid.*, **52**, 1027 (1937).

[4] ψ is a conveniently selected function satisfying certain conditions (see section II).

[5] G. Horvay, *Phys. Rev.* **55**, 70 (1939).

[6] D. H. Weinstein, *Proc. Nat. Acad. Sci.* **20**, 529 (1934).

[7] J. K. L. MacDonald, *Phys. Rev.* **46**, 828 (1934).

[8] A. F. Stevenson, *Ibid.*, **53**, 199 (1938).

[9] G. Horvay, *Ibid.*, **56**, 214 (1939).

[10] The word "admissible" implies that the function satisfy the prescribed boundary conditions.

[11] J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik*, p. 74 (1934).

[12] One arrives at a useful generalization of the methods discussed in this paper by introducing parameters into the wavefunction ψ. For example in the function $\psi(u)$ we may replace the independent variable u by σu where σ is the so-called scale parameter; or we may consider the linear combinations $\psi = \sum_{0}^{l}\sigma_i x_i$ of a set of functions φ_i. With suitable modifications the method can be also adapted to these cases. One constructs the admitted regions ${}^{r}U_s[x_1, \ldots, x_s; \sigma_\alpha, \sigma_\beta, \ldots]$ for all values of the parameters. The envelope of the boundaries of these regions delimits their common portion, the new admitted region, which is no longer dependent on the parameters. Such procedures permit in many cases a considerable reduction in the size of the original admitted regions without necessitating too much extra labor.

THE NON-LINEAR BOUNDARY VALUE PROBLEM OF THE BUCKLED PLATE

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Introduction.—The problem of the buckling of thin plates under thrust in the plane of the plate with large deflections has become of increasing practical importance in recent years, because of the wide use of thin-walled structures, particularly in aircraft design. The mathematical formulation of the problem involves a pair of non-linear differential equations derived by v. Kármán.[4,7] Various writers[5,10,11,12,13,14,16,20] have attacked the problem using perturbations and the energy method in order to obtain solutions valid at least for a limited range of the ratio of applied pressure to the lowest buckling pressure. Such solutions are given in the main for the rectangular plate, which is the most important for the applications. An exact solution for this case valid for an unlimited range of the above ratio presents seemingly insurmountable mathematical difficulties. For the circular plate (1), however, we have obtained rigorous solutions valid for an unlimited range. Important and interesting conclusions can then be drawn, which retain at least qualitative significance for the rectangular plate. In the latter case, for example, an edge effect has been observed experimentally[2,3,8,12,13,18,19] and made the basis of the definition of an "effective width" by v. Kármán.[8] In our case we find such an edge effect to be a natural part of the solution, and explain it by means of an asymptotic treatment in analogy with the Prandtl boundary layer theory. For the flow of a viscous fluid around an obstacle, cf., e.g., reference 15.

1. *Mathematical Formulation.*—We consider a plane circular plate with the radius R and thickness h simply supported and subject to a constant radial thrust at the edge. We assume the deflection w and the radial membrane stress to depend only upon the distance r from the center of the plate.

Quantities which we call stresses here are to be understood as stresses divided by the modulus of elasticity E. Quantities which are barred or which carry the subscript o refer to their values at the edge or center, respectively.

Upon introduction of the negative radial membrane stress p and the quantity $q = Rr^{-1}dw/dr$ the v. Kármán equations can be reduced to

$$Gp = \frac{1}{2}q^2, \quad \eta^2 Gq + pq = 0,$$

where $G = R^2 r^{-3} \frac{d}{dr} r^3 \frac{d}{dr}$,

$\eta = \gamma h/R$ the "slenderness ratio,"
$\gamma = [12(1 - \nu^2)]^{-1/2}$, and ν is Poisson's ratio.

The boundary conditions at the edge ($r = R$) are

$$r\,dq/dr + (1 + \nu)q = 0 \text{ and } p = \bar{p},$$

where the important quantity $\bar{p}$ is the prescribed pressure at the edge. Further we require p and q to be regular at the center ($r = 0$).

Of the constants R, h, E, ν, $\bar{p}$, which characterize the problem, only two, ν and the "thrust ratio" $\bar{p}/\eta^2$, are essential in the sense that deflection and stresses in two plates having the same values of these ratios differ only by constant multipliers. We found it convenient to take for ν the fixed value 0.32; thus $\bar{p}/\eta^2$ is the only constant to be varied.

2. *Perturbation Method.*—This method (2) consists in developing $\bar{p}$, p and q with respect to a parameter ϵ into a series

$$\begin{aligned} \bar{p} &= \bar{p}^0 + \epsilon^2\bar{p}^{(2)} + \epsilon^4\bar{p}^{(4)} + \ldots, \\ p &= p^0 + \epsilon^2 p^{(2)} + \epsilon^4 p^{(4)} + \ldots, \\ q &= \epsilon q^{(1)} + \epsilon^3 q^{(3)} + \ldots \end{aligned}$$

One then obtains a sequence of linear differential equations for p^0, $q^{(1)}$, $p^{(2)}$, $q^{(3)}$, ..., which can be solved successively, though with great labor. The quantity p^0 proves to be a constant identical with the lowest value of $\bar{p}$ for which buckling begins and is the same as that obtained by the linear theory; its value is $p^0 = 4.24\,\eta^2$.

We computed $q^{(1)}$ to $q^{(5)}$ and p^0 to $p^{(6)}$. We found that the convergence is satisfactory only in a small range of values of $\bar{p}/p^0$; namely, for q up to $\bar{p}/p^0 = 1.2, 1.8, 2.5$ if we include terms of the first, third and fifth order, respectively, and for p up to $\bar{p}/p^0 = 1.4, 2.2, 2.8$ if we include terms of the second, fourth and sixth order, respectively (3). For the other stresses the convergence is not quite as good. Nevertheless we find some qualitative features already in this range (cf. section 4).

3. *Method of Integration of S. Way.*—Another method is available for treating this problem, a method used by S. Way[21] for treating the bending of circular plates. It consists in setting up series for p and q in powers of r, assuming values for the first coefficient of each series, that is, for the values of p and q at the center of the plate. The values of p at the edge must then be determined by solving the transcendental equations resulting from a boundary condition.

With this method it is possible to obtain solutions for a much higher range of values of $\bar{p}/p^0$ with very much less labor in numerical computation than would be required by the perturbation method. We have carried out the calculations up to $\bar{p}/p^0 = 14.7$; for this it was necessary to compute thirty terms in each series. The method has its disadvantages, however. One wishes to obtain a fairly even distribution for the values of $\bar{p}/p^0$ and at

the same time one wants to know at least roughly where the solutions of the above-mentioned transcendental equations will be. In order to accomplish this with a reasonable amount of computation, it is necessary to make fairly accurate estimates of the values of p and q at the center. How to do this is not obvious, for the values of p and q change with increasing thrust ratio in a rather surprising way, e.g., the value of p at the center changes sign (cf. section 4). Our results obtained from the perturbation method (which requires no estimates in advance), proved to be an excellent guide for this purpose.

4. *Discussion of Results.*—It was to be expected that the membrane stress $(-p_0)$ at the center of the plate would decrease quickly, but it seems not to have been observed that this stress, originally a compression $(p_0 > 0)$, actually becomes a tension $p_0 < 0$. This transition occurs for $\bar{p}/p^0 = 1.57$. For $\bar{p}/p^0 = 2.5, 14.7$, the membrane stress is tension from the center up to $r/R = 0.63, 0.88$, respectively. It is remarkable that the ratio $p^0/\bar{p}$ of the membrane stress at the center to that at the edge attains a minimum, which has the value -0.76 and occurs for $\bar{p}/p^0 = 5.5$; for $\bar{p}/p^0 > 5.5$ this ratio rises again. The circumferential membrane stress at the edge $(-\bar{p}_c)$ increases very rapidly with increase of $\bar{p}/p^0$ and, for $\bar{p}/p^0 = 14.7$, reaches the value $-12.3\bar{p}$. The radial bending stress $(\bar{p}_b)$ at the outer surface of the plate first attains its maximum at the center, but from the transition value $\bar{p}/p^0 = 1.57$ the maximum point shifts toward the edge; for example, when $\bar{p}/p^0 = 2.5, 14.7$, it occurs at $r/R = 0.60, 0.87$, its value there is 1.35, 37.0, times its value at the center, respectively, and in both cases nearly equal to the circumferential membrane stress at the edge. For a fixed slenderness ratio η, the bending stress at the center has a maximum $10.22\,\eta^2$ for $\bar{p}/p^0 = 3.7$; with further increase of the thrust ratio it tends to zero. The deflection curves flatten out perceptibly in the middle portion of the plate with increasing thrust ratio, though this effect is perhaps less pronounced than might have been expected.

5. *Limit State and Asymptotic Solution.*—We discuss here what occurs when $\bar{p}/p^0$ tends to infinity, or, what is the same thing, when the thrust ratio $\bar{p}/\eta^2 \to \infty$. This passage to the limit may be achieved in different ways; for example, we might take a fixed plate (η fixed) and allow $\bar{p}$ to increase indefinitely, or hold $\bar{p}$ fixed and consider plates with slenderness ratios η tending to zero.

In order to determine the limit state one may proceed by letting $\eta \to 0$ in the original differential equations. One then obtains

$$Gp = \frac{1}{2}q^2, \qquad pq = 0,$$

which have as sole regular solution $q = 0$, $p =$ constant. One is then tempted to fix this constant by setting it equal to the prescribed value $\bar{p}$

at the edge. This means that in the limit state there would be a hydrostatic compression throughout the plate. Such a procedure corresponds to the treatment of laterally loaded clamped "sheets" by Hencky,[6] cf. also [1], and it may well be legitimate in such a case where no edge thrust is prescribed. In our case, however, wrong results would be obtained, since the stress in the interior of the plate tends rather to a state of hydrostatic tension. That is, the above solution $q = 0$, $p = \text{const.}$ remains valid, but the constant must be determined in a different manner.

There is a transition from tension in the interior to the prescribed compression at the edge which takes place in a narrow strip, the breadth of which tends to zero with increasing $\bar{p}/\eta^2$. In order to investigate the limit state mathematically, we treat this edge effect in a manner similar to that used for Prandtl's boundary layer theory. We stretch the plate so that the width of the edge strip does not shrink to zero by introducing the new variable

$$\beta = (1 - r/R)\bar{p}^{1/2}/\eta, \qquad 0 \leq \beta \leq \bar{p}^{1/2}/\eta.$$

We take $P = p/\bar{p}$, $Q = \eta q/\bar{p}$ and the limit state is then characterized by the differential equations

$$d^2P/d\beta^2 = \frac{1}{2}Q^2, \qquad d^2Q/d\beta^2 + PQ = 0, \qquad 0 \leq \beta < \infty,$$

by the boundary conditions $P = 1$, $\overline{dQ/d\beta} = 0$ (for $\beta = 0$), and regularity for $\beta = \infty$.

The solution of this "asymptotic" non-linear boundary value problem can be obtained with relative ease, and furnishes limit values for the essential quantities (4). Some of the results and a comparison with our results for finite edge thrust are given in the following table (5).

1	$\bar{p}/\bar{p}^0$	1	1.57	2.5	5.1	14.7	∞
2	$p_0/\bar{p}$	1	0.00	−0.52	−0.76	−0.67	−0.47
3	$p_{b0}/\bar{p}$	0	3.92	3.11	1.55	0.26	0.00
4	$\eta p_{b\ max}/\bar{p}^{3/2}$	0	1.52	1.34	1.25	1.21	1.11
5	$\eta p_c/\bar{p}^{3/2}$	0.49	1.02	1.28	1.46	1.55	1.61
6	$\eta q/\bar{p}$	0	1.19	1.41	1.53	1.59	1.61
7	$w_0/R\bar{p}^{1/2}$	0	1.90	2.50	2.96	3.23	3.29

This table indicates that the solutions really do approach the asymptotic solution and, in fact, come quite near to it for rather low values of $\bar{p}/\bar{p}^0$. The convergence toward the asymptotic solution becomes even more striking if one plots curves for the above quantities using all available calculated numbers. Strictly speaking, the limit values −0.47 and 0 indicated in rows 2 and 3 refer to the inner edge of the boundary layer and not to the center of the plate. The constant $-0.47\bar{p}$ is the one which should be taken

for fixing the edge value of p if one proceeds in the manner of Hencky as indicated above (6). Hence we are justified in identifying these limit values with those for the center.

We note that the passage to the limit may be accomplished in such a way that slope and stresses remain small enough to insure the validity of the v. Kármán equations: from row 6 (which presents the most critical case) we see that it is only necessary to let η and $\bar{p}$ tend to zero in such a way that $\bar{p}/\eta$ remains sufficiently small.

Another objection on physical grounds to the passage to the limit may well be raised. It is likely that there is a value of the thrust ratio for which the buckled state in its turn becomes instable, and what might be called a "second buckling" sets in (7). If one were to pursue the "second buckling" into the non-linear range it is likely that it would eventually also become instable and a "third buckling" ensue, and so on. Various experimenters, working with rectangular plates, have found that the plates became wrinkled near the edges.[2,3,5,12,13,18,19] We feel that such wrinkles are to be explained as the result of a buckling of order higher than the first—an explanation which, as far as we know, had not yet been explicitly advanced.

(1) The problem of the bending of a circular plate under lateral pressure has been solved by S. Way[21] by an exact method for a rather large range of applied load. We have employed this method in part in our own work (cf. section 3).

(2) The convergence of this process is insured by a theorem of E. Schmidt. Cf., e.g.[9].

(3) Marguerre[13] mentions that his calculations for the rectangular plate, which implied two perturbations may perhaps be valid up to $\bar{p}/p^0 = 20$; Timoshenko[20] applies his Ritz method with three constants for $\bar{p}/p^0 = 59.7$ and mentions that an additional constant would be necessary only if $\bar{p}/p^0 > 50$. In our simpler case we find that values of $\bar{p}/p^0$ of these magnitudes cannot be treated with even three perturbations. In fact such values of $\bar{p}/p^0$ lead to solutions already in the asymptotic range (cf. section 5).

(4) In the same manner, we can easily obtain the asymptotic solutions for the clamped plate under edge moment treated by Way.[21] We hope to investigate at a later date asymptotic solutions for plates loaded laterally and without prescribed edge thrust with the view to check the soundness of the Hencky method.

(5) We note in addition the formula $(\tilde{p}_c/\eta^2)/(\bar{p}_c/\eta^2)^{2/3} \longrightarrow 1.07$, where $\tilde{p}_c = R^{-1}\int_0^R p_c dr$ is the average circumferential stress. This formula presents an analogy to the formula of v. Kármán,[8] in which, however, the power $1/2$ takes the place of the power $2/3$.

(6) This differs from Prandtl's boundary layer theory: there the limit state in the interior furnishes a quantity which must be used for determining the solution of the boundary layer equations.

(7) Rough calculations indicate that the "second buckling" may take place for $\bar{p}/p^0 \sim 15$. This is a question which we expect to investigate at a later date.

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GROUPS WHICH CONTAIN TEN OR ELEVEN PROPER SUBGROUPS

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The groups which separately contain less than ten proper subgroups, that is, subgroups which are neither the identity nor the entire group, were recently determined in these PROCEEDINGS, **25**, 367–371 (1939). If an abelian group involves exactly ten proper subgroups its order cannot be divisible by more than three distinct prime numbers and when it is divisible by three such numbers one of its factors is the square of a prime number and it is cyclic. There is then one and only one such group for every possible set of three distinct prime numbers. There is a cyclic

group of order $p_1p_2^5$ and also one of order $p_1^2p_2^3$, p_1 and p_2 being distinct prime numbers, which has exactly ten proper subgroups. If a non-cyclic abelian group contains exactly ten proper subgroups and its order is divisible by two distinct prime numbers its order is of the form $9p$, where p is any prime number besides 3, and the subgroup of order 9 is non-cyclic. If a prime power cyclic group contains exactly ten proper subgroups its order is of the form p^{11}, p being an arbitrary prime number. As there is no prime power non-cyclic abelian group which contains exactly ten proper subgroups *there are exactly five abelian groups which separately contain exactly ten proper subgroups.*

We proceed to prove that there is no prime power non-abelian group which contains exactly ten proper subgroups. It is known that every prime power non-cyclic group contains a non-cyclic invariant subgroup of order p^2 whenever this prime number p is odd.[1] This subgroup contains $p + 1$ subgroups of order p. The invariant subgroup of order p^3 of the group which contains this non-cyclic subgroup of order p^2 involves operators of order p^2 since the number of the proper subgroups of G is supposed to be ten. When $p > 3$ this subgroup of order p^3 contains more than ten proper subgroups and when $p = 3$ it contains eight proper subgroups. Hence a prime power non-cyclic group cannot contain exactly ten proper subgroups except possibly when its order is a power of 2 since a group of order 3^4 clearly contains more than two proper subgroups which are not found in the given subgroup of order p^3.

If the order of the group is of the form 2^m and it contains exactly ten proper subgroups $m > 4$ since it is well known that none of the fourteen possible groups of order 16 contains exactly ten proper subgroups. Moreover, all the groups of order 16 which do not contain an operator of order 8 contain more than ten proper subgroups and each of the four non-abelian groups of order 32 which separately involve a cyclic subgroup of order 16 contains more than ten proper subgroups. As a non-cyclic group of order 16 contains at least nine proper subgroups there is no non-abelian group whose order is of the form 2^m which contains exactly ten proper subgroups. That is, every non-abelian group which contains exactly ten proper subgroups has an order which is divisible by at least two distinct prime numbers.

Such an order could not be divisible by as many as four distinct prime numbers, for if it were divisible by four such numbers the largest of these would be at least as large as 7. If it were 7 or larger the Sylow subgroup whose order would be divisible by this prime number would be invariant since a group of degree 8 whose order is divisible by 7 contains more than ten subgroups. The Sylow subgroup whose order is a power of the largest prime number which divides the order of the group G would transform each of the other Sylow subgroups in G into itself since it would transform

at least one such subgroup into itself because G involves only ten proper subgroups. As similar remarks apply to the Sylow subgroups whose order is divisible by next to the largest prime number which divides the order of G it has been proved that if a non-abelian group contains exactly ten proper subgroups its order is divisible by at most three distinct prime numbers, and if its order is divisible by three such numbers the Sylow subgroup whose order is divisible by the largest prime number which divides the order of the group is invariant and transforms into itself each of the other Sylow subgroups contained in the group.

We shall now prove that there is only one system of non-abelian groups which is composed of groups which separately involve exactly ten proper subgroups and whose orders are divisible by three distinct prime numbers in each case. In such a group the Sylow subgroup whose order is divisible by the largest prime number which divides the order of the group is of prime order since the group contains only ten proper subgroups, and the group is the direct product of this Sylow subgroup and the group generated by its other Sylow subgroups. The latter group is necessarily the symmetric group of order 6 since it must involve exactly four proper subgroups. Moreover, every direct product of this symmetric group and a group of order p, where p is a prime number which exceeds 3, contains exactly ten proper subgroups. That is, *the direct products of the symmetric group of order 6 and a group of prime order larger than 3 give all the non-abelian groups which separately involve exactly ten proper subgroups and whose orders are separately divisible by as many as three distinct prime numbers.*

It remains to determine the non-abelian groups which separately involve exactly ten proper subgroups and whose orders are divisible by exactly two distinct prime numbers. The Sylow subgroups whose orders are divisible by the larger of these prime numbers are again invariant under the group G but they could not transform into themselves the Sylow subgroups whose orders are divisible by the smaller of these two prime numbers unless G is a Hamiltonian group. If it is not Hamilton an the larger of these prime numbers cannot exceed 7. If it is 7 then G is obtained by either establishing a 7,2 isomorphism between the dihedral group of order 14 and the cyclic group of order 4 or by establishing a 7,3 isomorphism between the semi-metacyclic group of order 21 and the cyclic group of order 9. When the larger prime number which divides the order of G is 5 then G is formed by establishing a 5,4 isomorphism between the dihedral group of order 10 and the cyclic group of order 8. Finally, when the larger prime number is 3 then G is formed by establishing a 3,8 isomorphism between the symmetric group of order 6 and the cyclic group of order 16.

When G is a Hamiltonian group its order is not divisible by 16 since the Hamiltonian group of order 16 contains seventeen proper subgroups. It

is therefore the direct product of the quaternion group and the group of an arbitrary odd prime order. By combining these results we obtain the following theorem: *The total number of groups which separately contain ten proper subgroups is eleven, composed of five abelian ones and six non-abelian ones.* Each of the five abelian groups is composed of an infinite system of groups of similar structure. Only two of the six non-abelian groups are composed of infinite systems of groups of similar structure. The remaining 4 non-abelian groups are individual groups of definite orders.

The abelian group which contains exactly eleven proper subgroups is the cyclic group of order p^{12}, p being any prime number. The fact that the order of such a group s a power of a prime number results from the fact that $11 + 2$ is a prime number since the total number of the subgroups of an abelian group is the product of the total number of the subgroups in its Sylow subgroups. It could not have as many as three invariants because a group of the type 1^3 contains more than eleven subgroups. It could not have two invariants of which one is a prime number since 11 is a prime number. Its invariants could not be two composite numbers since such a group involves more than eleven subgroups. It therefore results that it must be cyclic and of order p^{12}.

If a non-abelian group contains exactly eleven proper subgroups it cannot be Hamiltonian and hence it involves non-invariant subgroups. Its order could not contain as many as four distinct prime factors for the same reasons as those which relate to the groups which contain exactly ten proper subgroups. In the present case the order could not be divisible by as many as three distinct prime numbers because 11 is a prime number. The order of such a group could not be a power of a single prime number for similar reasons to those stated in connection with the groups which contain exactly ten proper subgroups. Hence it results that if a non-abelian group contains exactly eleven proper subgroups its order is divisible by two and only two distinct prime numbers.

The larger of these two prime numbers cannot be as large as 7 because G is solvable. For the same reason it cannot be as large as 5. The group G cannot contain more than one Sylow subgroup whose order is a power of 3 because if it contained four such subgroups it would be isomorphic with the tetrahedral group. The number of Sylow subgroups whose order is a power of 2 could not exceed 3 and hence these Sylow subgroups would be transformed either according to the alternating group of degree 3 or according to the symmetric group of this degree. As this is impossible it has been proved that *the cyclic group of order* p^{12} *is the only group which contains exactly eleven proper subgroups.*

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LINEAR OPERATIONS AMONG SUMMABLE FUNCTIONS

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Let S be an arbitrary space, $\mathfrak{E}$ a fixed Borel field of subsets E in S with $S \in \mathfrak{E}$ and $\alpha(E)$ a fixed non-negative completely additive measure function defined over $\mathfrak{E}$. Letting $\mathfrak{E}_B$ denote those $E \in \mathfrak{E}$ for which $\alpha(E) < \infty$, the triad $[S, \mathfrak{E}, \alpha]$ is a *system* if there exists a *decomposition of* S, i.e., if there are a denumerable number S_i of disjoint elements of $\mathfrak{E}_B$ such that $S = \Sigma_i S_i$. When the system $[S, \mathfrak{E}, \alpha]$ and the exponent q are fixed $L^q(S)$ will denote[2] the Banach space (B-space) composed of those numerical functions $\phi(s)$ defined over S, measurable with respect to $\mathfrak{E}$, and having $||\phi|| = (\int_S |\phi(s)|^q d\alpha)^{1/q} < \infty$ for $1 \leqq q < \infty$ and $||\phi|| = \text{ess.sup.}\ |\phi(s)| < \infty$ for $q = \infty$. When $q < \infty$ the adjoint space of $L^q(S)$ is $L^{q'}(S)$ and $L^q(S)$ is separable if and only if $\mathfrak{E}_B$ is separable under the metric dist. $(E, E') = \alpha(E - E') + \alpha(E' - E)$.

Confining our attention in this note entirely to linear operations U sending $L(S)$ into a B-space X, we wish to communicate some results concerning the representations of U under certain restrictions on either U or X. These results yield general representations for various types of operators with range in $L^q(T)$, where $[T, \mathfrak{F}, \beta]$ is a second system. When S and T are bounded real intervals and α and β are Lebesgue measure, previous representations have been given by Dunford,[3] Gelfand,[4] Vulich,[5] Kantorovitch and Vulich[6] and others[7] for the general linear and general completely continuous[8] operation mapping $L(S)$ into $L^q(T)$, $1 \leqq q \leqq \infty$. The methods used by these authors have primarily depended, however, on the differentiation of real-valued or B-space-valued functions. Here, using the same general approach but different methods, we have extended these results to the case of an abstract S; in addition *weakly completely continuous*[9] operations are considered. By means of these representation theorems further light is cast upon certain kinds of operators. In particular a uniform mean ergodic theorem for weakly c.c. operators in $L(S)$ can be stated and applied to a well-known type of Markoff process to yield results somewhat sharper than those heretofore known.

Let $x(s)$ be defined to a B-space X from almost all of S. Calling E_0 a *null set* if $\alpha(E_0) = 0$, $x(s)$ is *essentially defined to* $Y \subset X$ if $x(S - E_0) \subset Y$ for some null set E_0. If $x(s)$ is essentially defined to a (weakly) compact set it is *essentially* (*weakly*) *compact-valued*. Finally, the measurable essentially bounded functions $x(s)$ form a class denoted by $A^\infty(S)[X]$. If $x(.) \in A^\infty(S)[X]$ the B-space integral[10] $\int_S x(s)\phi(s)d\alpha$ exists for each

$\phi(.) \in L(S)$ and defines a linear operation U from $L(S)$ to X having for its norm $|U| = \text{ess.sup.}||x(s)||$.

1. *General Operations to Restricted Spaces.*—The following theorem, which is basic for our representations of general operations on $L(S)$, may be considered as an extension of the Radon-Nikodym theorem[11] to set functions in an adjoint space.

THEOREM 1. *Let* $\mathrm{x_E}$ *be defined from* $\mathfrak{E}_B$ *to* $\mathrm{X} = [\mathrm{x}]$ *where* X *is the adjoint of* $\mathrm{Y} = [\mathrm{y}]$. *Suppose that* (i) Y' *is a separable linear subset of* Y, (ii) $\mathrm{y} \in \mathrm{Y}'$ *and* $\mathrm{E}' \in \mathfrak{E}_B$ *imply that* $\mathrm{x_E(y)}$ *is completely additive and absolutely continuous over* $\mathfrak{E}(\mathrm{E}')$,[12] *and* (iii) *the total variation* ν_{E}' *of the set function* $\sigma_{\mathrm{E}} \equiv \sup_{\mathrm{y}} \frac{1}{||\mathrm{y}||}|\mathrm{x_E(y)}|$, $\mathrm{y} \in \mathrm{Y}'$, $\mathrm{y} \neq 0$, *is finite over each* $\mathrm{E}' \in \mathfrak{E}_B$. *Then there exists an* $\mathrm{x_s}$ *on* S *to* X *such that* $\mathrm{x_E(y)} = \int_{\mathrm{E}} \mathrm{x_S(y)\,d\alpha}$ *and* $\nu_{\mathrm{E}} = \int_{\mathrm{E}} ||\mathrm{x_s}||\,\mathrm{d\alpha}$ *are true for* $\mathrm{E} \in \mathfrak{E}_B$ *and* $\mathrm{y} \in \mathrm{Y}'$.

When $E' \in \mathfrak{E}_B$ and $y \in Y'$ the set functions ν_E and $x_E(y)$ are completely additive and absolutely continuous over $\mathfrak{E}(E')$. For each sequence $\{y_j\}$ in Y' the decomposability of S and the Radon-Nikodym theorem then imply that $\phi_0(s)$, $\{\phi_j(s)\}$ exist such that $\nu_E = \int_E \phi_0(s)d\alpha$ and $x_E(y_j) = \int_E \phi_j(s)d\alpha$ hold for every $E \in \mathfrak{E}_B$. Hence $|\phi_j(s)| \leqq \phi_0(s)||y_j||$ holds uniformly in j a.e. in S. Taking $\{y_j\}$ additive and dense in Y' and applying a lemma of Doob[13] or a well-known moment theorem it follows that for almost every s an element x_s exists in X such that $||x_s|| \leqq \phi_0(s)$ and $x_s(y_j) = \phi_j(s)$. By the Hahn-Banach theorem it may be supposed that $||x_s|| = \sup_y |x_s(y)|$, $y \in Y'$, $||y|| = 1$. The denseness of $\{y_j\}$ and Lebesgue's convergence theorem, together with simple variation inequalities, lead to the desired conclusions.

Theorem 1 yields abstract representations for operations to X when (1) X is the adjoint of a separable space, (2) X is a separable adjoint space or (3) X has a weakly compact unit sphere.[14] In case (2) the general linear operation U is $U(\phi) = \int_S x(s)\phi(s)d\alpha$ where $x(.) \in A^{\infty}(S)[X]$ and under (3) this is the general separable[15] operation to X. The general linear operation $U(\phi) = x_\phi$ has the form $x_\phi(y) = \int_S x_s(y)\phi(s)d\alpha$ when (1) holds. In all three instances $|U| = \text{ess.sup.}\,||x_s||$ and $x(.)$ is essentially unique for each U. Case (1) has this corollary: if either $\mathfrak{E}_B$ or $\mathfrak{F}_B$ is separable then U is linear to $L^{\infty}(T)$ if and only if $U(\phi) = \int_S K(s, t)\phi(s)d\alpha$ where $K(s, t)$ is measurable and essentially bounded over $S \times T$; moreover $|U| = \text{ess.sup.}\,|K(s, t)|$.[16] Case (3) is applicable when $X = L^q(T)$, $1 < q < \infty$ (Theorem 8).

In representing the general linear operation from $L(S)$ to $L(T)$ our methods restrict T to be a finite or infinite real interval with end-points c and d, $-\infty \leqq c < d \leqq \infty$. Let $BV(T)$ be those numerical functions $\nu(t)$ which are each of bounded variation over the interior of T. Each $\nu(.)$ in $BV(T)$ has the limits $\nu(d - 0)$ and $\nu(t + 0)$ existing for every $t \in T$. If $\nu(.)$ in BV-

(T) satisfies the additional conditions that $\nu(c+0)=0$, $\nu(t+0)=\nu(t)$ for $t \epsilon T$ and $\nu(d-0)=\nu(d)$ if $d \epsilon T$, we say that $\nu(.)$ is in $C^*(T)$. With a suitable norm $BV(T)$ is a B-space in which $C^*(T)$ forms a closed linear manifold equivalent to the adjoint of $C(T)$, $C(T)$ being the separable space of real functions $\gamma(t)$ which are continuous over T and have $\gamma(c+0)$ and $\gamma(d-0)$ existing finitely. From Theorem 1 there can be drawn with the aid of several lemmas

THEOREM 2. *Any linear operation* $U(\phi)=\nu_\phi$ *from* L(S) *to* C^*(T) *has a representation*

$$U(\phi)=\nu_\phi=\int_s K(s,t)\phi(s)d\alpha, \qquad \phi(.) \epsilon L(S), \tag{2.0}$$

where (2.1) $K(s,.) \epsilon C^*(T)$ *for each* s, (2.2) $K(.,t) \epsilon L^\infty(S)$ *for each* t, (2.3) K *is measurable and* ess.sup. $|K(s,t)|<\infty$, (2.4) ess.sup.$_s$ $[\mathrm{Var}_t K(s,t)] \equiv M<\infty$, *and* (2.5) *the two mixed* (*Stieltjes and Lebesgue*) *integrals* $\int_c^d \gamma(t)d_t\{\int_S K(s,t)\phi(s)d\alpha\}$ *and* $\int_S\{\int_c^d \gamma(t)d_tK(s,t)\}\phi(s)d\alpha$ *exist and are equal whenever* $\gamma(.) \epsilon C(T)$ *and* $\phi(.) \epsilon L(S)$. *Conversely if* K(s,t) *satisfies the conditions* (2.6) $K(s,.) \epsilon C^*(T)$ *for almost every* s, (2.7) K(s,t) *is measurable in* s *for every* t *in a dense subset of* T, *and* (2.8) ess.sup. $|\int_c^d \gamma(t)d_tK(s,t)|<\infty$ *for each* $\gamma(.) \epsilon C(T)$, *then* K *has the additional properties* (2.2)–(2.5) *and the operation* U *of* (2.0) *is defined and linear from* L(S) *to* C^*(T). *The norm of* U *is the constant* M *of* (2.4).

Since the indefinite integral $\int_c \phi(t)dt$ is in $C^*(T)$ for each $\phi(.)$ in $L(T)$ Theorem 2 yields[17]

THEOREM 3. *Any linear operation* U *to* L(T) *has a representation of the form*

$$U(\phi)=\frac{d}{dt}\int_s K(s,t)\phi(s)d\alpha, \qquad \phi \epsilon L(S), \tag{3.0}$$

where $\int_S K(s,t)\phi(s)d\alpha$ *is absolutely continuous in* t *for each* ϕ *and* K *has properties* (2.1)–(2.5); *the norm of* U *is the constant* M *of* (2.4). *Conversely, if* K(s,t) *satisfies* (2.6)–(2.8) *the operation* U *of* (3.0) *is defined and linear to* L(T) *and* K *has the properties* (2.2)–(2.5); *here* $|U| \leqq M$ *with equality holding if* $\int_E K(s,t)d\alpha$ *is absolute continuous in* t *for each* $E \epsilon \mathfrak{E}_B$.

Some operations from $L(S)$ to $L(T)$ cannot be written as $U(\phi)=\int_s H(s,t)\phi(s)d\alpha$ where H is measurable. However, if U is linear to $L(T)$ and among its representations (3.0) there is one with K measurable and $K(s,.)$ essentially defined to a separable set in $BV(T)$ then U can be written as $U(\phi)=\int_s H(s,t)\phi(s)d\alpha$ where H is measurable, $|U|=$ ess.sup.$_s$ $\int_T |H(s,t)|dt$ and for almost every s the equality $\frac{d}{dt}K(s,t)=H(s,t)$ holds a.e. in T.

2. *Restricted Operations to General Spaces.*—With the aid of known results in the theory of convex sets in B-spaces the following proposition can be established.

THEOREM 4. *If* $x(.) \epsilon A^{\infty}(S)[X]$ *the integral* $\int_S x(s)\phi(s)d\alpha = U(\phi)$ *defines a separable operation* U *that has its range in* X, *takes weakly compact sets into compact sets, and has* $|U| = \text{ess.sup.}||x(s)||$. *Moreover,* U *is (weakly) c.c. if and only if* x(.) *is essentially (weakly) compact-valued.*

Theorem 4 combined with a differentiation theorem[18] furnishes

THEOREM 5. *Let* S *be a finite or infinite* n*-dimensional Euclidean interval and let* α *be Lebesgue measure. Then* U *is a (weakly) c.c. operation to* X *if and only if* $U(\phi) = \int_S x(s)\phi(s)d\alpha$ *where* x(.) *is an essentially (weakly) compact-valued element of* $A^{\infty}(S)[X]$.

Using these two theorems it is rather simple to discuss inclusion relationships among classes of operations mapping $L(S)$ into a given X. For example let $S = [0, 1]$ and $X = L(S)$. Taking a bounded but not weakly compact sequence $\{x_n\}$ in $L(S)$ and defining $x(s)$ to be x_n for $s \epsilon J_n = (1/n - 1, 1/n]$, $U(\phi) = \int_S x(s)\phi(s)d\alpha$ takes weakly compact sets in $L(S)$ into compact sets in $L(S)$ but it is not weakly c.c. On choosing $\{x_n\}$ to be weakly compact in $L(S)$ but not compact, U becomes weakly c.c. but not c.c. An example due to von Neumann[19] exhibits an operation in $L(S)$ defined by a bounded kernel yet not c.c.: the construction amounts to choosing the functions $x_n(s)$ to be uniformly bounded over S (and therefore weakly compact) but not compact as a set in $L(S)$. The Haar orthonormal set is such a sequence.[20]

Theorem 4 enables us to state the following uniform mean ergodic theorem for certain operations in $L(S)$.

THEOREM 6. *Let* $T = S$, $\mathfrak{F} = \mathfrak{E}$ *and* $\beta = \alpha$, *and suppose* $U(\phi) = \int_S x(s)\phi(s)d\alpha$ *where* x(.) *is an essentially weakly compact-valued element of* $A^{\infty}(S)[L(T)]$. *These conclusions can be reached:* (I)(i) *there is a measurable kernel* $K_1(s,t)$ *defined over* $S \times T$ *such that* $K_1(s,.) = x(s)$ *in* L(T) *for almost every* s *and* $U(\phi) = \int_S K_1(s,t)\phi(s)d\alpha$ *for* $\phi \epsilon L(S)$, (ii) $|U| = \text{ess.sup.}||x(s)|| = \text{ess.sup.} \int_T |K_1(s,t)|d\beta$, (iii) U *is separable, weakly c.c. and takes weakly compact sets into compact sets, and* (iv) U^n *is c.c. for* $n \geq 2$ *and hence the fixed points of* U *form a closed linear manifold* Y *having a finite dimension number* N.

Suppose in addition that the norms $|U^n|$ *are bounded. Then* (II)(i) $\lim_{m,n \to \infty} |V_m - V_n| = 0$ *where* $V_m = \frac{1}{m}\Sigma_1^m U^i$, (ii) *the limit operation* $V = \lim V_m$ *is c.c. from* L(S) *to* L(S), (iii) $V^2 = V$ *and* $VU^n = U^nV = V$ *for* $n = 1, 2, \ldots$, *and* (iv) Y *is the set of fixed points for* V *and* $Y = V(L(S))$.

Setting $x_0(s) \equiv V(x(s))$ *we also have* (III)(i) $x_0(.) \epsilon A^{\infty}(S)[Y]$, $V(\phi) = \int_s x_0(s)\phi(s)d\alpha$, *and* $|V| = \text{ess.sup.}||x_0(s)||$, (ii) $x_0(s) = U^n(x_0(s)) = V(U^n(x(s))) = V(x_0(s))$ *for every* n *and almost all* s, (iii) $x_0(s) = \Sigma_1^N \mu_i(s)\phi_i$ *where* $\phi_i \epsilon Y$, $||\phi_i|| = 1$, $\mu_i \epsilon L^{\infty}(S)$, *and* $\int_s \mu_i(s)\phi_j(s)d\alpha = \delta_{ij}$, *the Kronecker delta, and* (iv) *in* $L^{\infty}(S)$ *the closed linear manifold* Y^* *determined by the* μ_i*'s is the set of fixed points for both* U^* (*the adjoint of* U) *and the c.c. adjoint* V^* *of* V, *and* Y^* *has dimension* N.

The following statements are true concerning the measurable kernels $K_0(s,t) \equiv \Sigma_1^N \mu_i(s)\phi_i(t)$, $K_1(s,t)$, *and* $K_n(s,t) \equiv \int_S K_{n-1}(s,s')K_1(s',t)d\alpha$, $n = 2, 3, \ldots$: (IV)(i) $V(\phi) = \int_S K_0(s,t)\phi(s)d\alpha$ *and* $U^n(\phi) = \int_S K_n(s,t)\phi(s)d\alpha$ *for* $n = 1, 2, \ldots$, (ii) *for almost every* s *the equality* $K_0(s,t) = \int_S K_{n+1}(s,s')K_0(s',t)d\alpha = \int_S K_0(s,s')K_n(s',t)d\alpha$ *holds for* $n = 0, 1, 2, \ldots$ *and for almost all* t, *and* (iii) $K_n(s,.)$ *is essentially weakly compact-valued in* $L(T)$ *for* $n = 1$ *and essentially compact-valued for* $n \neq 1$.

Finally there is a null set E_0 *such that* (V)(i) $\lim_{n\to\infty} \int_T |K_0(s,t) - \frac{1}{n}\Sigma_1^n K_i(s,t)|d\beta = 0$ *uniformly over* $S - E_0$, (ii) $\lim_n \frac{1}{n}\Sigma_1^n \int_F K_i(s,t)d\beta = \int_F K_0(s,t)\,d\beta$ *uniformly over* $(S - E_0) \times \mathfrak{F}$ *and* (iii) $\lim_n \int_T |\int_S \phi(s)K_0(s,t)\,d\alpha - \int_S \frac{\phi(s)}{n}\Sigma_1^n K_i(s,t)d\alpha|d\beta = 0$ *uniformly over those* ϕ *in* $L(S)$ *having* $\int_S |\phi(s)|d\alpha \leqq 1$.

Theorem 6 is a consequence of Theorem 4, a mean ergodic theorem of Kakutani[21] and Yosida,[22] and some lemmas on convex sets. An immediate corollary of Theorems 5 and 6 is that if S is a Euclidean interval Theorem 6 holds for any operation U that is weakly c.c. in $L(S)$.

Let $[S, \mathfrak{E}, \alpha]$ and $[T, \mathfrak{F}, \beta]$ be two systems with $T = S$, $\mathfrak{F} \subset \mathfrak{E}$, $\beta = \alpha$ and $\mathfrak{F}_B$ separable; let P be the essentially non-negative elements of $L(S)$ and $P_1 = P[||\phi|| = 1]$. Suppose $\pi(s, F)$ is defined over $S \times \mathfrak{F}$ and has these properties: (1) $\pi(s, F)$ is measurable in s for each $F \epsilon \mathfrak{F}_B$, (2) for each s $\pi(s, F)$ is non-negative and completely additive over $\mathfrak{F}$ and $\pi(s, T) = 1$ and (3) if $F_1 \supset \ldots \supset F_n \supset \ldots$ and $\beta(\lim_n F_n) = 0$ then $\lim_n \beta(s, F_n) = 0$ uniformly over S. As Doob has shown,[23] $\pi(s, F)$ is a set of conditional probability functions for a certain class of temporally homogeneous Markoff processes. For each s there is an element $x(s)$ of P_1 whose indefinite integral is $\pi(s, F)$; moreover, this function $x(.)$ is a weakly compact-valued element of $A^\infty(S)$-$[L(T)]$. Hence if $U(\phi) = \int_S x(s)\phi(s)d\alpha$ Theorem 6 furnishes

THEOREM 7. *Conclusions* (I)–(V) *inclusively of Theorem 6 are all true for this operation* U. *In addition we have* (VI)(i) $\pi(s, F) = \int_F K_1(s,t)d\beta$ *for every* s *and every* F, $K_n(s, t) \geqq 0$ *and* $\int_T K_n(s, t)d\beta = 1$ *for each* s *and* n, (ii) $|U^n| = |V| = 1$ *for* $n = 1, 2, \ldots$, (iii) *if* $P_1 \ni \phi$ *then* $P_1 \ni U^n(\phi)$ *and* $V(\phi)$ *for* $n = 1, 2, \ldots$, (iv) N *is a positive integer and* (v) *the elements* $\phi_1, \ldots, \phi_N$ *of* Y *may be chosen in* P_1.

Theorem 7 constitutes an extension of certain results obtained by Doob,[24] Kakutani and Yosida[25] and others.

The following statements are true concerning operations to $L^q(T)$, S and T being arbitrary. An analogue for $q = \infty$ exists but is omitted here.

THEOREM 8. *For a fixed* $q < \infty$ *suppose* $K(s, t)$ *has these properties:* (i) $K(s, t)$ *is measurable,* (ii) $\int_T |K(s, t)|^q d\beta < \infty$ *for almost all* s *and* (iii). ess.sup. $|\int_T K(s,t)\psi'(t)d\beta| < \infty$ *for every* $\psi'(.) \epsilon L^{q'}(T)$. *Then* (iv) ess.sup.$_s$

$(\int_T |K(s,t)|^q d\beta)^{1/q} \equiv C < \infty$, (v) $U(\phi) = \int_S K(s,t)\phi(s)d\alpha$ *is a separable operation from* $L(S)$ *to* $L^q(T)$, (vi) U *takes weakly compact sets into compact sets,* (vii) *the norm of* U *is the constant* C *of* (iv) *and* (viii) *if* $q > 1$ U *is weakly c.c. Statement* (viii) *is not always true when* $q = 1$, *but if in this case* K *satisfies besides* (i)–(iii) *the condition* (ix) *a null set* E_0 *exists such that*

(a) $\lim_{\beta(F)=0} \int_F K(s,t)d\beta = 0$ *uniformly on* $S - E_0$, *and*

(b) *for each decomposition* $\{T_i\}$ *of* T $\lim_{m\to\infty} \sum_1^\infty |\int_{T_i} K(s,t)d\beta| = 0$ *uniformly on* $S - E_0$,

then U *is weakly c.c on* $L(S)$ *to* $L(T)$. *For* $1 \leqq q < \infty$ U *is c.c. if the kernel* K *satisfies besides* (i)–(iii) *the condition* (x) *for some null set* E_0' *the set* $K(s,.)$, $s \in S - E_0'$, *is compact in* $L^q(T)$. *Conversely, if* U *is an arbitrary separable operation from* $L(S)$ *to* $L^q(T)$, $1 < q < \infty$, *a kernel* K *exists satisfying* (i)–(viii). *If* S *is Euclidean and* U *is weakly c.c. to* $L(T)$ *there is a* K *satisfying* (i)–(ix). *If* S *is abstract and* U *is c.c. to* $L^q(T)$, $1 \leqq q \leqq \infty$, *there is a* K *for which* (i)–(x) *hold.*

Let S be the real axis and $T = [0, 1]$. Suppose $K(s)$ is periodic over S with period 1 and $K(.) \in L^q(T)$, $1 \leqq q < \infty$. From Theorem 8 it follows that $U(\phi) = \int_S K(s-t)\phi(s)ds$ is defined and c.c. from $L(S)$ to $L^q(T)$ and $|U| = (\int_0^1 |K(t)|^q dt)^{1/q}$.

Finally there is

THEOREM 9. *Suppose* $x(.) \in A^\infty(T)[X]$. *If* K(s,t) *satisfies conditions* (i)–(iii) *and* (ix) *of Theorem 8 for* $q = 1$ *then* $U(\phi) = \int_T x(t)\{\int_S K(s,t)\phi(s)d\alpha\}d\beta$ *is a c.c. operator from* $L(S)$ *to* X.

One of the corollaries to Theorem 9 has this for a special case: if $S = T = W = [0, 1]$ and all measures are Lebesgue's and if $K(s, t)$ and $K'(t, w)$ are each bounded and measurable then $U(\phi) = \int_S \phi(s)\{\int_T K(s, t)K'(t, w)dt\}ds$ is c.c. from $L(S)$ to $L^q(W)$, $1 \leqq q < \infty$. For $q = 1$ this reduces to a recent result of Servint.[26]

[1] Sterling Fellow.

[2] We shall write $L(S)$ for $L^1(S)$ and q' for the conjugate exponent of q.

[3] *Trans. Amer. Math. Soc.*, **40**, 474–494 (1936).

[4] *Recueil Math.* (Moscou), **4**, 235–284 (1938).

[5] *Ibid.*, **2**, 275–305 (1937).

[6] *Comp. Math.*, **5**, 119–165 (1937); *Ibid.*, **5**, 430–432 (1938).

[7] For further references see Kantorovitch and Vulich, loc. cit.

[8] The phrase "completely continuous" will be abbreviated to "c.c."

[9] An operation is weakly completely continuous (Kakutani and Yosida) if it is linear and takes bounded sets into weakly compact sets. A set is weakly compact if every infinite subset contains a subsequence converging weakly to an element of the space.

[10] According to the equivalent definitions of T. H. Hildebrandt, S. Bochner and N. Dunford.

[11] Or as an extension of Gelfand's Theorem 2, loc. cit., p. 261.

[12] $\mathfrak{E}(E')$ is the Borel field of all elements of $\mathfrak{E}$ which are subsets of E'.

[13] *Duke Math. Jour.*, **4**, 752–774 (1938).

[14] See Gelfand, loc. cit.

[15] An operation U from Y to X is separable if it is linear and $U(Y)$ is separable in X.

[16] See Gelfand and Kantorovitch and Vulich, loc. cit.

[17] See Gelfand, Vulich, and Kantorovitch and Vulich, loc. cit.

[18] B. J. Pettis, *Duke Math. Jour.*, **5**, 254–269 (1939).

[19] Einar Hille and J. D. Tamarkin, *Ann. Math.*, **35**, 445–455 (1934). Also see J. Servint, *Compt. Rend. URSS*, **18**, 255–257 (1938) and K. Yosida, Y. Mimura and S. Kakutani, *Proc. Imp. Acad. Tokyo*, **14**, 359–362 (1938).

[20] Servint's example uses the Haar set.

[21] *Proc. Imp. Acad. Tokyo*, **14**, 295–300 (1938).

[22] *Ibid.*, **14**, 292–294 (1938).

[23] *Trans. Amer. Math. Soc.*, **44**, 87–150 (1938).

[24] Loc. cit.

[25] *Proc. Imp. Acad. Tokyo*, **14**, 333–339 (1938).

[26] Loc. cit. Also see the joint paper of Yosida, Mimura and Kakutani.

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